Quantum Geometry of the Ising Model

Marit Sandstad

Thesis submitted for the degree of
MASTER OF PHYSICS

Departement of Physics
University of Oslo

May 29, 2009

Abstract

We introduce some modern mathematical and theoretical tools in 2-dimensional physics, and apply them to the Ising model. We rederive some well-known results, but also some new properties of this important model. All the tools presented here have applications for beyond the Ising model.

We explore several aspects of conformal field theory, proving, analysing, and testing Zamolodchikov's C-theorem. We explore finite size effects in critical and non-critical systems on the cylinder and the torus, and discuss the implications of modular invariance. Using BCFT, we explore the implications for theories with a boundary, and look at an interesting relationship between BCFTs and non-critical theories for integrable models, which indicates that there is a deep link between conformal symmetry and the symmetry of integrability. Finally we explore the holographic projection of critical and off-critical models which relates flat 2-dimensional models to 3-dimensional gravity.

Acknowledgements

This thesis was written under the guidance of Carsten Andrew Lütken. I am very grateful to him for the opportunity to dive into this extremely interesting realm of physics. His foresight and ideas for what to do have been wonderful and has made the writing of this work an incredible learning experience that has been at the same time more focused and widespread than I was able to realise till quite late in its making.

Jaakko Nissinen has been a useful discussion partner and collaborator on the material for chapters 9 and 10.

Further, this work was written at the department of physics at the University of Oslo in the theoretical physics group. I am very grateful to both the department and the group for providing a hospitable and comfortable place to study, read, write, and calculate.

Helle Frisak Sem has read through several drafts for this thesis and helped in improving the language and taking out misprints. Without her help the former would have been worse and the latter more numerous.

I am also indebted to Stian Grenborgen for both moral and technical support throughout this process.

This thesis is dedicated to my five grandparents.

Oslo, May 2009
Marit Sandstad

Contents

Notation

Throughout this text we will work with $k = \hbar = c = 1$.

We will make repeated use of the following acronyms: RG = Renormalization group, QFT = Quantum field theory, CFT = Conformal field theory, BCFT = boundary conformal field theory, AdS = Anti de Sitter (space).

The standard mathematical shorthand notations iff = “if and only if”, and s.t. = “such that”, will also be used from time to time.

$\Re x$ will denote the real part of x, $\Im x$ will denote the imaginary part of x.

Chapter 1

Introduction

Ceci n'est pas une pipe

Inscription on the René Magritte painting *La trahison des images* (1928-1929) depicting a pipe.

Just as a painting or picture of an object never *is* the actual object, a theory or model of a physical system never *is* the physical system. When asked about his painting René Magritte argued that though it looked like a pipe, his pipe could not be stuffed, a property that all pipes should have. In the same way it would be foolish to believe that a model of a physical system could truly embody every property of the system.[1]

What then should we as physicists aim for in constructing a model? We should aim for an effective, workable model, that is, a model that describes as many of the most important properties of the actual physical system as possible. However, when several options exist, the simplest theory that describes our system in a satisfying manner should be chosen.

This means that we should carefully sort out irrelevant properties from relevant ones. For instance, whether you choose to call a certain variable ϕ or ψ is probably not relevant. On the other hand, the dimension of the geometry on which the theory is formulated is normally very relevant.

Our model should also make testable predictions, and its range of applicability should be specified, that is a list of reasonable assumptions that should be fulfilled for the theory to be valid. For instance, no single model should be expected to adequately explain the phenomena at every energy scale. The theory of RG-flows has taught us a lot about how this works, both unifying models in the same universality class, and dividing theories that are not in the same universality class, separating them perhaps by what their effective degrees of freedom are.

[1] Properties that we might in fact not know about in constructing the theory.

1.1 Research Questions

In this thesis we will make a survey of useful modelling techniques and frameworks in describing the physics of two dimensions. We will explore such questions as:

1. What are the properties of effective field theory descriptions of critical 2-dimensional systems?
2. What restrictions are imposed on the flows in RG-space of 2-dimensional theories, and how can we calculate these flows?
3. What is the effect of compactifying one or two of the dimensions?
4. What happens to a critical theory in 2-dimensions if the 2-dimensional geometry has boundaries? What sensible boundary conditions can we impose and how do they restrict the field theory content?
5. Which infinite symmetries are found away from criticality and are they somehow connected to the symmetry of the critical theories from which they flow?
6. Can 2-dimensional theories capture the dynamics of the 3-dimensional spacetimes more adequately than Magritte could capture the properties of a 3-dimensional pipe on a 2-dimensional canvas?

These questions will be exemplified by that simplest 2-dimensional pet model, the Ising model, and its critical effective field theory, the Majorana fermions.

Most of the quantitative results we obtain for the Ising model are well-known, but some will not be, perhaps because a modern review of the Ising model illuminated by the modern mathematical tools that have appeared in the past two decades, in the wake of string theory, has yet to appear.

As for the general results and frameworks, most have been known and cherished for decades while others have been developed very recently and are still being explored at the time of writing of this thesis.

1.2 Outline

This thesis is divided into three main parts. Chapters 2, 3 and 4 are background theory chapters introducing the Ising model and some of the tools we will be using. Chapters 5 through 10 are the main part of the work, where calcutlations and proofs are carried out. Chapter 11 concludes and summarises the work, focusing on the future possiblities. There is also an extensive appendix listing properties of important functions, stating and proving useful theorems, as well as reviewing some mathematics that may be unfamiliar to most physicists.

Chapter 2 gives a very short and by no means exhaustive description of the Ising model.

Chapter 3 reviews some geometry and topology, with a special focus on certain 2- and 3-dimensional topological spaces. We will give a thorough introduction to several 2-dimensional structures focusing especially on the torus. A review of the 3-dimensional AdS space and its relative the BTZ black hole is also included in this chapter.

Chapter 4 gives an introduction to conformal field theory, and thereby a partial answer to question 1 above, since the effective field theories of 2-dimensional critical theories are CFTs.[2] Starting with a short description of universality and what we will come to call critical phenomena, gives us the appliction of this theory to the statistical mechanical realm. A short introduction to the renormalisation group ties this description up with the quantum field theory description, which will be used in many calculations.

Chapter 5 gives a detailed proof of Zamolodchikovs C-theorem, the answer to question 2. The chapter tries to explain some of the consequences of this theorem and also goes on to a short exploration into the work on generalizations of it, how they are made up, why they might not work, and in fact cannot work in the arbitrary dimension and geometry case.

Chapters 6 through 8 contain calculations of quantities in the field theoretic version of the Ising model, the (massive) Majorana fermions.

Starting with calculations on the plane in chapter 6, we see the application of Zamolodchikov's C-theorem. Going through the direct calculations we end up by listing properties of the limiting massless theory.

Moving the theory onto the cylinder in chapter 7, calculating finite size effects in the vacuum energy, starts off our survey of finite size effects and the answer to question 3.

Then moving on to the torus in chapter 8, we repeat the calculation of vacuum energy and look into the effects of modular invariance, thereby giving the rest of the answer to our third question.

In chapter 9 we briefly review a curious similarity between theories on a geometry with boundary at criticality, and an off-critical theory, a property linked to that of integrability. This combines the answers to questions 4 and 5 in a very intriguing manner.

In chapter 10 we turn to holography, describing how the critical Ising model is the dual of a certain 3-dimensional gravity theory. We also explore how this holographic description may prove useful in non-critical cases and hence the outlook of the answer to our final question.

Finally, in chapter 11 we summarise our answers and results. We will

[2]Actually, the starting point of the answers to questions 3 and 4 can also be found here.

also briefly speculate on future uses of the tools and frameworks discussed here, trying to separate real restrictions from the mere concrete but non-essential qualities of the Ising model, and the direction in which these tools can be taken further.

Chapter 2

The Ising Model: A Short Introduction

> I would like to suggest that there is no such thing as a left or right. There is only an up or down ...
>
> Ronald Reagan in his 1964 *A Time for Choosing* speech.

One of the most famous statistical models, the Ising model is the model of spins on a lattice (Di Francesco et al., 1997).

The spins at each lattice point may be directed up or down. The Hamiltonian gets contributions from the coupling between neighbouring spins and all the spins may also be coupled to an external magnetic field.

If we let a spin up at lattice site $\vec{x}$ be denoted by $\sigma_{\vec{x}} = +1$, and spin down at $\vec{x}$ be denoted by $\sigma_{\vec{x}} = -1$, the Hamiltonian is given by:

$$H = -J \sum_{\langle \vec{x}, \vec{x}' \rangle} \sigma_{\vec{x}} \sigma_{\vec{x}'} - \sum_{x} B_{\vec{x}} \sigma_{\vec{x}} \tag{2.1}$$

where $J \in \mathbb{R}$ is a coupling constant, $B_{\vec{x}}$ is the external magnetic field, and the sum over $\langle \vec{x}, \vec{x}' \rangle$ is a sum over neighbouring spins (Ravndal, 1995). In the applications in this thesis the external field will always be zero and there will therefore be a discrete symmetry between spin up and spin down.

In one dimension this model has no phase changes, and it is fairly trivial to solve. In dimensions $d > 2$ the model has not been solved. However, the 2-dimensional Ising model is soluble. In this thesis, we will use the Ising model on a 2-dimensional square lattice as our prototype, to which we apply many interesting new geometrical tools in order to rederive some well-known, and some not so well-known, results of this 2-dimensional Ising model.

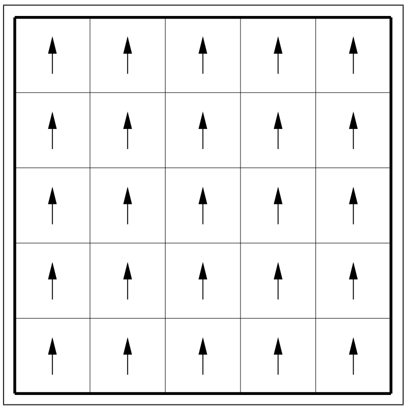

Figure 2.1: This figure shows an example of a low temperature limit of the Ising model on a 5 by 5 lattice. It may be considered as an infinitesimal piece of the full thermodynamic lattice. All other parts will look the same with all spins pointing in the same direction.

2.1 The 2-Dimensional Ising Model on a Square Lattice

In two dimensions, the Ising model has a second order phase transition at non-zero temperature, which was first proven by Lars Onsager in 1944 (Baxter, 1982) .[1]

In the zero temperature limit, the model is in a state where all spins point in the same direction, as shown in figure 2.1. With respect to the symmetry between spin up and spin down, this ground state is not symmetric. Rather the symmetry is broken by this ground state.

In the infinite temperature limit the spins will be completely randomly distributed. An example of a piece of such a lattice is shown in figure 2.2. Here the symmetry of the hamiltonian under exchange of spin up and spin down is manifest in the state itself.

The two different regimes are in fact different phases. In between the two, there is a non-trivial phase transition caused by spontaneous symmetry breaking. The theory at this critical temperature is called the critical Ising model.

The thermodynamic quantity with a discontinuity at the Ising model critical point is the magnetic moment, and this is a second order derivative of the partition function, so the transition is of second order.[2]

The magnetic moment is also called the local order parameter of the

[1]The 2-dimensional Ising model can be solved on several other lattices too, see (Baxter, 1982).

[2]More on transitions and critical behaviour can be found in chapter 4.

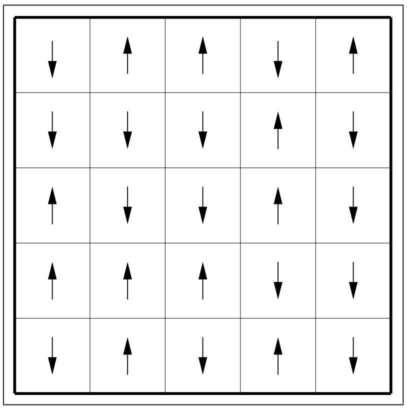

Figure 2.2: This figure shows an example of a high temperature limit of the Ising model on a 5 by 5 lattice. It may be considered as an infinitesimal piece of the full thermodynamic lattice. The spins point up and down in no particular pattern, at random. The other parts of the lattice will not be identical, but they will also have spins pointing in random directions. The direction of each spin in this lattice was decided by a computer random generator. Though such a generator does not produce truly random numbers, we may assume them to be sufficiently random for our purposes here.

phase shift. Such a parameter has a zero expectation value in the ordered or symmetric phase and a nonzero expectation value in the disordered or nonsymmetric phase. We realise that the ordered phase is the high-temperature phase, whereas the disordered one is the low-temperature phase.

The critical point was first found by Kramers and Wannier in 1941 (Baxter, 1982) by means of what is now called Kramers-Wannier duality.

We will not explain the details of the Kramers-Wannier transformation here, rather we will give a short and simplified presentation and list the result. A full treatment can be found in for instance (Baxter, 1982).

We start by defining the coupling constant $K = \beta J$. We can then make a mapping between high and low energy couplings by means of the transformation:

$$K \to K_* \qquad \sinh 2K_* = \frac{1}{\sinh 2K} \tag{2.2}$$

It can be shown that the free energy $\mathcal{F}(K_*)$ for the transformed phase is given by:[3]

$$\mathcal{F}(K_*) = \mathcal{F}(K) + \ln(\sinh 2K) \tag{2.3}$$

[3]See for instance (Baxter, 1982).

Since the two phases are mapped into one another by this relation, the existence of a critical point at $K = K_c$ implies that there will also be a critical point at $K_* = K_c$. Thus if at this value $K \neq K_*$, there must be at least two critical points. If we assume there is only one critical point it must be at $K = K_*$, i.e.:

$$\sinh 2K_c = 1, \qquad K_c = 0.44068679$$

It can be proved analytically or by numerical experiments that this is in fact a critical point and that it is also the only one.

We shall come back to descriptions of and results for the Ising model throughout this thesis.

Chapter 3

Geometrical Preliminary: Useful Low-Dimensional Structures and a Taste of Topology

> A topologist is one who doesn't know the difference between a doughnut and a coffee cup.
>
> John Kelley

In this chapter we will go through some topology and geometry that we will find useful in the rest of this work. We first give a brief reminder of some concepts from general topology, as well as an indication of how these concepts will be used here. We assume that the reader has some familiarity with the subject.[1]

Since the calculations in this thesis are mainly conducted on 2-dimensional and 3-dimensional geometries, we will give a short introduction to the 2- and 3-dimensional structures in question.

3.1 Some Useful Topological Concepts

Topology is in essence the most general theory for studying continuous functions. In mathematics, topology is essentially that which is invariant under continuous functions or deformations. This is the sense in which a coffecup is equivalent to a doughnut. A continuous deformation is called a homeomorphism.

[1]Two useful source are (Nash and Sen, 1983) and (Nakahara, 2003).

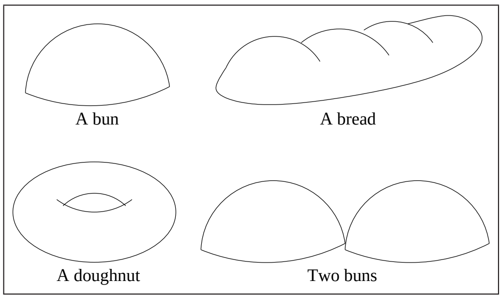

Figure 3.1: Starting from a bunshaped piece of dough shown on the upper left, we may continuously deform it to a loaf of bread by rolling it out. To make a doughtnut from it however, we must either rip a hole in the bun or glue the two ends of the bread together. Both these processes are discontinuous. To get two buns from the bread we must tear the bun or bread into two separate pieces. We can also make them from the doughnut by tearing it apart in two places. None of these processes are continuous.

The formal defintion of topology is given in terms of open (and closed) subspaces, but we shall not give these definitions here.[2] [3]

The picture of a cookie dough space is perhaps useful. Anything that we can make by simply deforming the dough is topologically the same, but whenever we start deforming it in a noncontinuous way, e.g. we poke a hole in the dough, tear it apart or glue it together, we change the object's topology. Hence a bun and a loaf of bread are topologically equal, whereas the doughnut, or two buns, are not topologically equivalent to each other or to the loaf of bread (see figure 3.1).

Some properties are preserved under continuous deformations, and these are called topological invariants. A couple of important examples:

Connectedness, which is defined by its opposite. A space is disconnected if it can be split in two open disjoint nonempty subspaces. When using this concept here, we will in fact mean pathwise connectedness which means that given two points in the space there is a path inside the space

[2] The author learned her topology in the classical mathematics way, as given in (Holm, 2004). There are, however, several good presentations of the material written for physicists, for instance (Nash and Sen, 1983) and (Nakahara, 2003). The presentations for physicists are shorter and more efficient and are more to the point for the physicist. The mathematical presentation is more thorough and offers deeper insight, but in doing so, it nessescarily spends more time on what most physicists would consider useless pathologies.

[3] A word of caution: in physical applications, most spaces have much more structure than pure topology. Normally we will look at spaces invariant under differentiable maps for instance. Physicists are prone to use the word topology to denote whatever is left when all dynamics are taken away. This is normally more than topology. In this work we will for instance assume all spaces to be manifolds. For more on manifolds, see for instance (Spivak, 1970, 3rd edition 2005)

connecting the two.[4] In dough language; a bread is connected, two buns are not.

Compactness means that if the space is covered by some collection of open subspaces, there exists a finite subcollection of these open subspaces that still cover the space. In a finite-dimensional euclidian space, all closed and bounded objects are compact. Therefore, everything we make of dough is compact. However $\mathbb{R}^n$ is not compact and neither is any open subset of it.

Orientability is essentially a continuous choice of bases around any closed curve in the space. Hence the cylinder is orientable, the Möbius band is not.[5]

The number of properties that are topological are infinite. We can always devise new ones by combining the old concepts. However, there are classifications that try to break it down to the minimal amount of information needed. Algebraic topology is the subject where this is done. Homology and cohomology break most of it down, however, there are also so-called secondary cohomological data.[6]

3.2 2-Dimensional Geometry

When first learning vector geometry, we may easily jump to the conclusion that the dimension of a space is nothing more than the number of vectors in a basis, and that in fact, 1-dimensional, 2-dimensional, and 456-dimensional spaces are in fact no more conceptually different than one bucket of sand is from two buckets of sand or fourhoundredandfiftysix buckets of sand.

Although this is in many ways true, there are never the less properties that may be special to a certain dimension or up to some dimension. For instance if you take out a point from a 1-dimesional space, the space may become disconnected. However, taking out a point of a higher dimensional space does not cause this behaviour.

2-dimensional space is exceptional for many reasons. For one, it is completely classified. As for compact 2-dimensional objects without a

[4] Pathwise connectedness implies connectedness, however there are several examples showing that the opposite is not true. However, these examples are rather pathological. When restricting ourselves to manifolds, connectedness and pathwise connectedness is in fact the same thing.

[5] A cylinder can be made by glueing two opposite ends of a strip of paper together. A Möbius band is made in the same way, but before the ends are glued together the paper should be twisted around 180°.

[6] We will not go into much detail of cohomology and homology. However, the reader can find more in (Hatcher, 2002) (for a mathematical treatment) and (Nash and Sen, 1983) and (Nash, 1991, fourth edition 2003) and (Nakahara, 2003) (for the physics view point).

boundary[7], they are completely defined by orientablity and their genus[8]. 2-dimensional spaces are also interesting beacuse of their relation to the complex plane. In principle we can use complex coordinates to describe our 2-dimensional space.[9]

In this work we will mainly be interested in a few select 2-dimensional objects. The non-compact objects that interest us will be the plane $\mathbb{R}^2$, and the infinite cylinder. The only object with boundary we consider is the annulus. We will describe these briefly below. Then we will describe the compact 2-dimensional object that will be of main interest to us; the torus.

3.2.1 The plane and the cylinder, and the annulus

In this section we describe the plane, the cylinder, and the annulus in brief.

The plane may be parametrised by two cartesian coordinates x and y, two polar coordinates r and θ, or two complex coordinates z and $\overline{z}$[10].

The most important things to know about the plane is its invariances. The plane is invariant under rotation, translation and global rescaling.

In topology, there is a theorem that states that all topological spaces may be compactified by the addition of one point. This is called the one point compactification. The usual one point compactification of the plane is by adding the point at infinity. This one point compactified plane is isomorphic to the sphere. We may see this through stereographic projection from the sphere to the plane. Stereographic projection is shown in figure 3.2. For an in depth explanation, see for instance (Bak and Newman, 1997).

The cylinder is made from the plane by taking a finite piece of one of the two coordinates x or y, hence obtaining an infinite strip. Then we identify the edges and get a cylinder.[11]

Assuming that the y-coordinate is the one being compactified to have a value between 0 and l, a nice mapping from the plane to the cylinder in complex coordinates is given by:[12]

$$z \to w \qquad \text{where} \qquad w = \frac{l}{2\pi} \ln z \tag{3.1}$$

More insight to the cylinder is found by viewing this map in light of the polar coordinates on the plane. With this map a two point compactification of the cylinder is identified with the one point compactified plane. The radius is mapped to the cylinder height, taking the unit circle to the height

[7] Objects with boundaries are more complex, however they are related to objects without boundaries in several ways. Identifying the boundaries make them into other objects, however there may be several ways of doing this for the same object.

[8] The genus is essentialy the number of holes in the surface.

[9] A simplified description of this identification can be found in Appendix II.

[10] See Appendix II for a brief review of this.

[11] A picture of this can be seen in figure 3.4.

[12] This mapping is given in (Cardy, 1990).

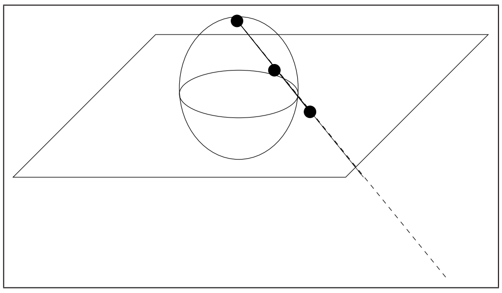

Figure 3.2: This figure shows stereographic projection from the (Riemann) sphere to the (complex) plane. The sphere is placed so that the plane intersects it along the equatorial circle. A point on the sphere is identified by a point on the plane by drawing a straight line from the north pole to the point. The point where this line intersects the plane is then the image of the original point on the sphere. The north pole is mapped to the point at infinity.

zero. The origin is mapped to the point at $-\infty$ and the point at ∞ is mapped to itself. A picture of this mapping can bee seen in figure 3.3.

The cylinder is still invariant under translations and rescalings of the infinite dimension, however, rotation invariance is no longer valid, since the directions are different. From the polar point of view we have seen above, the cylinder is not translationally invariant, since the origin is special, but it is rotationally invariant. However we view it though, some of the planar symmetries are broken.

The annulus is the only surface with a boundary that we will look at. It is essentialy a finite cylinder, however it may also resemble a flattened doughnut. If we identify the boundaries of the annulus, we get a torus, however, if we identify them with a twist we get a non-orientable surface called the Klein bottle. The annulus may also be subject to non-identifying boundary conditions, for instance free boundary conditions.

On the annulus, translational invariance and rescalings are broken, since a point on the boundary is very different from a point not on it, and the distance from edge to edge defines the annulus in question. However, translation along the periodic direction of the annulus is still a symmetry.

3.2.2 Classification of compact 2-surfaces

The classification theorem of compact surfaces states that a compact 2-dimensional surface without boundary is completely determined by orientability and its genus. In physical applications, non-orientable surfaces are seldom of interest to us.

Recall that the genus of the surface is given by the number of holes in

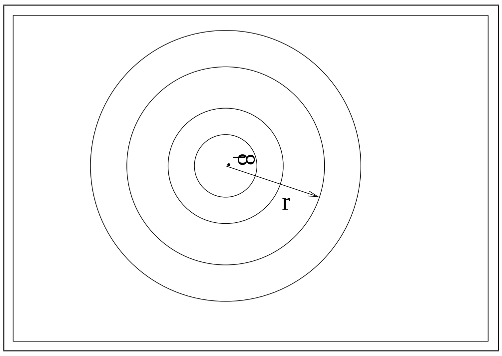

Figure 3.3: A picture of the mapping from the plane to the cylinder given by equation (3.1). The circle at radius r is mapped to the circular slice of the cylinder at height $\ln r$. The origin is mapped to the point at $-\infty$ whereas the point at ∞ in the plane is mapped onto the point at ∞ on the cylinder.

it. Or in the language of cohomology, the genus is given by the number of equivalence classes of closed curves. These form a group called the fundamental group, or π^1.

The sphere has no holes so the genus is zero, and all closed curves can be continuously deformed to points, so the group π^1 is trivial in this case.

On the torus, which has one hole and genus 1, $\pi^1 = \mathbb{Z} \times \mathbb{Z}$, the element (n, m) parametrising a curve that winds around the two boundary circles n and m times respectively. The signs of the numbers n and m denotes the direction of the curves. A surface of genus g has a group consisting of $2g$ copies of $\mathbb{Z}$.

Instead of denoting an orientable compact 2-surface by its genus, it is customary to define it by its Euler number[13] χ. For a compact 2-dimensional surface, the Euler number is given by:

$$\chi = 2 - 2g \tag{3.2}$$

3.2.3 Modular invariance and the torus

The torus is the compact 2-dimensional surface of Euler number $\chi = 0$. In this section we will give a short description of the torus and see how the modular group arises from natural assumptions of its structure. We will also describe the modular group, identifying it with $\mathrm{PSL}(2, \mathbb{Z})$.

The material in this section is quite standard and can be found in many good books on physics or mathematics. We feel that a thorough understanding of this material is of crucial importance in certain parts of

[13] For a proper definition of the Euler number and generalizations thereof, see (Nash and Sen, 1983).

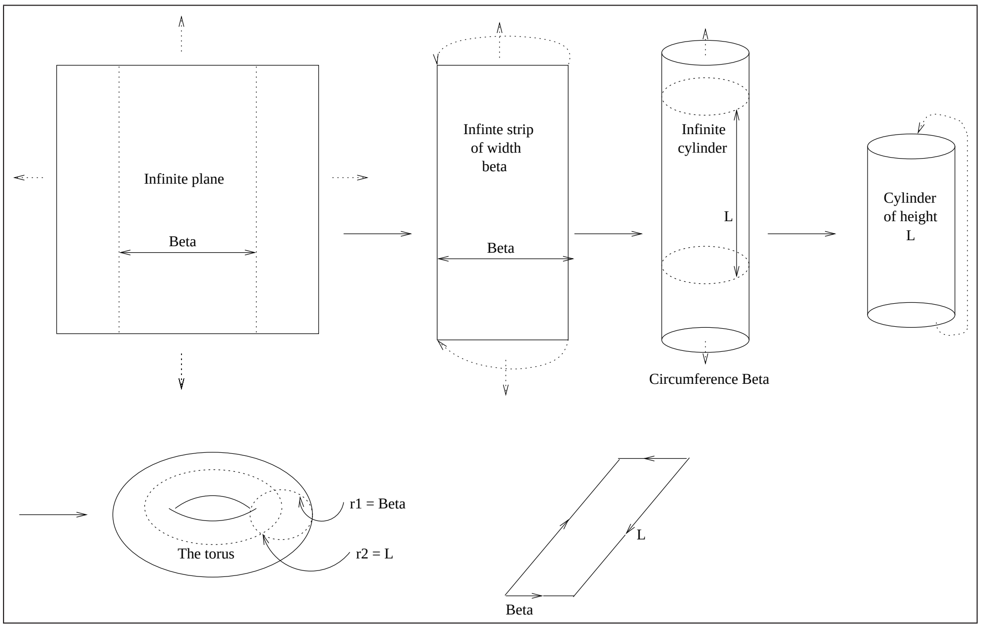

Figure 3.4: Here we see the construction of the torus from the plane through the cylinder in five steps. From left to right above we see the plane, the infinite strip, the cylinder and the annunlus. At last on the left below; the torus. The last part of the figure shows the unfolded torus, that is a parallellogram. The arrows indicate how the edges should be identified. Arrows pointing in opposite directions means no twist is required.

this work, and therefore include it here. This particular presentation of the material is largely inspired by its treatment in (Di Francesco et al., 1997).

The torus is most readily constructed by starting from the infinite cylinder. It is not difficult to realise that the cylinder can be obtained by taking a finite strip of the plane and identifying its edges.[14] We now repeat the process by cutting out a piece of the cylinder of finite length and identifying the two boundary circles. This process is shown i figure 3.4. In the figure we have also shown how a general torus looks when unfolded. If we place the torus in the complex plane by the identifiction of $\mathbb{R}^2$ with $\mathbb{C}$, the length L and width β can be arbitrary complex numbers. By rotation invariance in the plane we may assume β to be real and positive without loss of generality.

We can easily imagine different looking tori by varying the sizes of L and β and the relative angles between them. We can make the torus long and slim like a hula hoop or short and stubby like a doughnut.[15] But how

[14]See the three upper left figures in figure 3.4.
[15]See figure 3.5.

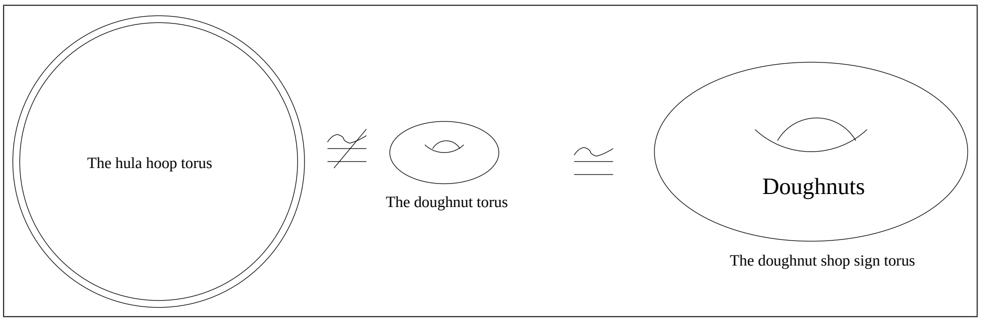

Figure 3.5: In this figure three different tori are shown. On the left we see a hula hoop shaped torus and in the middle a doughnut shaped torus. On the right we have simply increased the size of the doughtnut shaped torus just as we would do if we were making an attractive sign on top of our new doughnut shop. There exists a natural mapping between the two tori on the right. This mapping $f : z \to kz$ where k is some multiplicative constant is readily seen to be a conformal mapping, i.e. it is a holomorphic mapping.

different are these tori? Is it really natural to speak of different tori for every pair of complex numbers L and β? We have already deduced that we can assume β to be real, and this condition also leads us to the assumption that L must have a non-zero imaginary part. But are there stricter conditions? How can we tell truly different tori apart? Can we find equivalence classes of tori that are the same up to some transformation that should not affect our theory?

In the sense of topology, all tori are equal. That is, we can always stretch different tori continuously into one another. Alternatively we can consider homeomorphisms between the parallellograms from which they were constructed.

Look at all three tori in figure 3.5. Though none of them are the same, the two tori on the right are much more alike than any of them are to the hula hoop torus on the left. This is because the rightmost one is just an increased copy of the middle one.

In the sense of holomorphisms the doughnut and the hula hoop are not equal. We can not find holomorphic mappings between them, i.e. we need mappings that treat the y- and x-coordinates independently and not as a single $z = x + iy$-coordinate. This is not difficult to see, since the length and width of the hula hoop are very different, whereas the length and width of the doughnut tori are more comparable in sizes. However, if we simply increase the doughnut like on the right of figure 3.5, finding a viable holomorphic map is very easy.

Obviously not all tori that look different should give different physical theories. The doughnut and the doughnut shop sign have the same ratio between L and β and so we expect that a scale invariant theory sees them as equal.[16] They should then be in the same equivalence class since they are somehow the same torus. The hula hoop on the other hand had a different ratio. We should expect that the actual size of the torus should not matter in a theory with scale invariance, however, the ratio between the length and height should.

This argument is very heuristic, but the result is nevertheless the right one. The parameter which tells the tori apart is the ratio between the two radii

$$\tau = \frac{L}{\beta} \tag{3.3}$$

We hence put the tori into equivalence classes of equal τ. It now suffices to focus on one representative of each equivalence class.

However, this is not quite enough. If we let our representative torus be constructed from a parallellogram of sides 1 and τ ($\tau \in \mathbb{C}$ s.t. $Im(\tau) > 0$) with one corner in the origin, the τs are complex numbers in the upper half plane (denoted $\mathbb{H}$).

We want the theory to be invariant under the exchange of the notions of height and width. This means letting $\tau \to 1/\tau$. To keep the torus representative in the upper half plane we must add a minus sign to this transformation (i.e. keeping track of the orientation). So we want theories invariant under

$$\mathcal{S} : \tau \to -\frac{1}{\tau} \tag{3.4}$$

We also want theories invariant when τ is translated by an integer number of the period in β. This just means a transition between the representations of the cylinder from which we build the torus. Since all of these cylinders should be identified this change must be unphysical. So we also demand the theory be invariant under

$$\mathcal{T} : \tau \to \tau + 1 \tag{3.5}$$

or an arbitrary number of such translations.

Why invariance under $\mathcal{S}$ and $\mathcal{T}$ should be expected is also demonstrated in figure 3.6.

$\mathcal{S}$ and $\mathcal{T}$ together generate the modular group with elements

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1 \tag{3.6}$$

This can be identified with the group of 2-dimensional integer matrices of determinant 1, i.e. $\mathrm{SL}(2, \mathbb{Z})$, but since the transformation (3.6) is invariant

[16] As we shall see in section 4.3.1, a holomorphic mapping (or actually a meromorphic mapping), is a conformal mapping. In two dimensions, scale invariance implies conformal invariance (see chapter 4), so holomorphically equivalent tori should give the same physical theory in the scale invariant case.

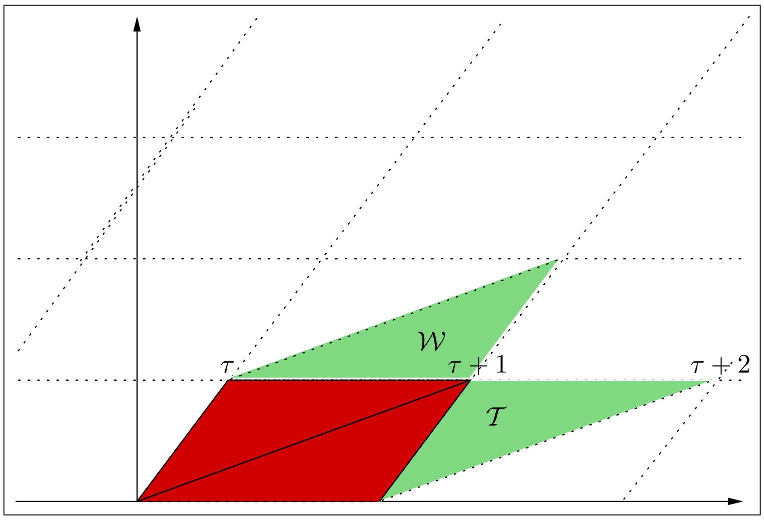

Figure 3.6: This figure shows the generators' actions on the torus. The dotted grid indicates the periodic tesselation of space created in the construction of the cylinder and thereafter the torus. The red torus is the original torus. The action of $\mathcal{T}$ on this yields the torus with half red half green to the right of the original one. We observe that the green piece is obtained by moving the missing red piece to the next cylinder representation. Exchanging height and length and repeating this trick we get the copy with the green part on top. This is the torus transformed by $\mathcal{W} = \mathcal{TST}$. Thus, since all the tessilations should be identified as one parallellogram in the torus construction, we see that modular invariance is a natural constraint.

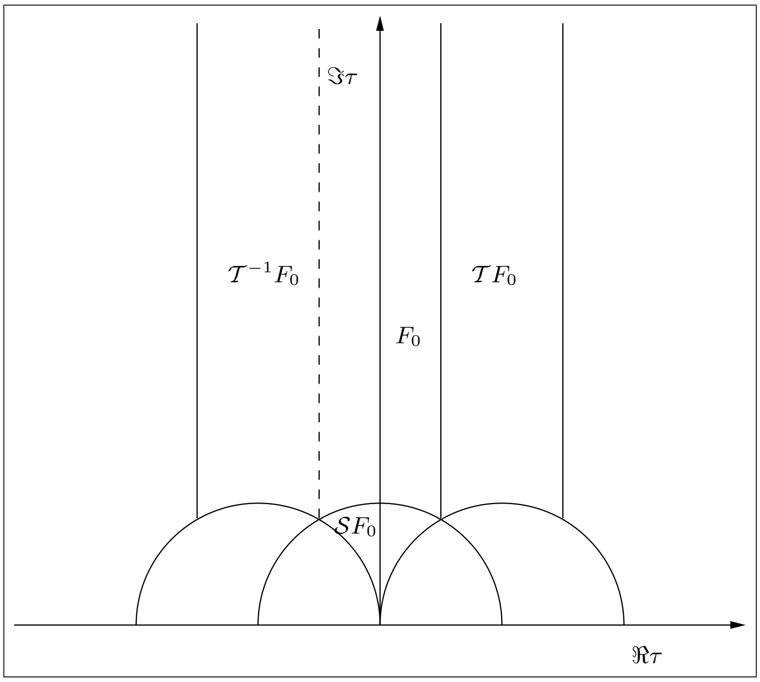

Figure 3.7: This figure shows the fundamental domain F_0 of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ in $\mathbb{H}$. For all points outside the fundamental domain there is a transformation in the modular gruop that takes a point in the fundamental domain to this point. By repeated use of the generators $\mathcal{T}$ and $\mathcal{S}$ we can make a tessilation of $\mathbb{H}$ covering the entire upper half plane with copies of the fundamental domain. The beginning of this tessilation is also visible here. The dotted line indicates that the fundamental domain is half open. This is also true for all its transformed copies, however, this is not shown here. Half the boundary between F_0 and $\mathcal{S}F_0$ should also be open, though this is not represented in this figure.

under the simultanous sign exchange of all four coefficients ($a \to -a, b \to -b, c \to -c, d \to -d$), we should mod out by $\mathbb{Z}_2$. This group is often called the projective special linear group over Z

$$\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2 \cong \mathrm{PSL}(2,\mathbb{Z})$$

We now see that the truly different tori are parametrized by τ in the area given by $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$. A representation of this area is given in figure 3.7 and will be called the fundamental domain.

Invariance under the modular group yields strong constraints on the theories allowed on the torus. The assumption that a theory should be consistent on the torus and scale invariant forces us to impose these strong constraints.[17] In an arbitrary theory these assumptions are far too strict.[18] However in the conformal case,[19] these assumptions are necessary as long as we want a consistent theory on the torus.

Certain combinations of the Jacobi θ functions are examples of modular invariants. The properties of these functions are reviewed in Appendix V.1 and we will use them to build a modular invariant partition function in chapter 8.

3.2.4 The flat torus

Recall from the complete classification of 2-dimensional surfaces above, that the torus has Euler number $\chi = 0$.

From the Gauss-Bonnet theorem,[20] we know that the curvature over the whole surface equals $2\pi\chi$. This means that the integral of the curvature on the torus is zero. This leads us to think that there might be flat representations of the torus. Though it is not possible to embed such a flat representation of the torus into 3-space,[21] it is possible to make such an embedding into 4-dimensional space.

It is possible to find a flat representation for all τ-tori, and we can hence always pick such a flat representation from the equivalence class without loss of generality. This will be of use, since we can then assume the torus we are working on to be flat.

The easiest picture of the flat torus for people not familiar with thinking in four dimensions (time not included) is the parallellogram shown to the right in the bottom of figure 3.4.

3.3 AdS Spacetime - a 3-Dimensional Structure

Anti-de Sitter (AdS) spacetime is a maximally symmetric spacetime with negative cosmological constant (Groen and Hervik, 2007), that is a space with constant negative curvature R. We will explore this spacetime because it has been claimed that such a spacetime is fully described by a theory on

[17] A scale invariant theory is a theory not affected by a local scaling of the the distances in spacetime. In chapter 4 we will discuss why theories at criticality are scale invariant.

[18] Normally the doughnut and the doughnut shaped sign are genuinely different, and height and width may not generally be interchanged.

[19] In chapter 4 we will also see that scale invariant theories in two dimensions are conformal. In general all conformal theories are scale invariant, however the opposite is not true in general.

[20] For a comprehensive explanation of the Gauss-Bonnet theorem see for instance (Jahren, 2007a).

[21] This is at least partially shown in (Jahren, 2007a).

its d-dimensional boundary and that this theory is a CFT.[22] [23]

Since its curvature is negative, AdS is the Minkowski equivalent of a maximally symmetric hyperbolic space.

In the context of holography it is customary to say that the AdS space is $(d+1)$- or $(d+2)$-dimensional. We will adopt the former notation, and so the metric of the $(d+1)$-dimensional AdS space in Poincaré coordinates is (Groen and Hervik, 2007):

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx_1^2 + \ldots + dx_{d-1}^2) \tag{3.7}$$

In this work we will mainly be interested in $d = 2$, so we will look more specifically at AdS_3. Here we can find global coordinates ρ, t and θ such that the metric becomes (Ryu and Takayanagi, 2006a,b):[24]

$$ds^2 = R^2(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\theta^2) \tag{3.8}$$

By solving the geodesic equations known from gravity (Groen and Hervik, 2007) and differential geometry (Jahren, 2007a), we can find the geodesics in this space.[25] We will use geodesic lengths in AdS_3 in our review of the correspondance between 3-dimensional gravity in AdS_3 and 2-dimensional conformal field theory in section 10.2 of chapter 10.

3.3.1 The BTZ black hole

Although the AdS_3 geometry is a solution of Einsteins vacuum equations in three dimensions with a negative cosmological constant, it is not the only one. In fact it is possible to insert a black hole at the origin.

This was first found by Bañados, Teitelboim and Zanelli (Bañados et al., 1992), whence this black hole is called the BTZ black hole.

In Schwarzschild coordinates its metric is given as (Birmingham et al., 2001):

$$ds^2 = -(-M + \frac{r^2}{R^2} + \frac{J^2}{4r^2})dt^2 + (-M + \frac{r^2}{R^2} + \frac{J^2}{4r^2})^{-1}dr^2 + r^2(d\phi - \frac{J}{2r^2}dt)^2 \tag{3.9}$$

where M and J are the mass and angular momentum of the black hole. These two properties completely define the BTZ black hole (Birmingham et al., 2001).

[22] This concept is called holography. More on holography will be given in chapter 10. For the moment we will refer the reader to the pedestrian introduction found in (Klebanov and Maldacena, 2009).

[23] More on CFT in chapter 4.

[24] A similar expression may be found in the general $(d+1)$-dimensional case if you exchange $d\theta$ with $d\Omega_{d-1}$ where $\int d\Omega_{d-1}$ is the area of a $(d-1)$-dimensional unit sphere.

[25] We will not do this in general since we will mainly be interested in some very special geodesics. However, it is not difficult to find the solutions analytically.

The black hole has an inner and outer Schwarzschild radius and it deforms the AdS_3 spacetime into a sort of solid torus (Birmingham et al., 2001). This is because the BTZ hole is actually obtained from AdS_3 by identifications of certain discrete subgroups of the isometry group (Bañados et al., 1993).

Exact solutions for the BTZ black hole are possible. For instance the geodesic equations are soluble. The time- and light-like geodesics were found in (Cruz et al., 1994). Space-like geodesics may also be found following their description.

In (Ryu and Takayanagi, 2006a) and (Ryu and Takayanagi, 2006b) it was claimed that the BTZ is the geometry in which to find a holographic description of a system at finite temperature.

In (Hotta et al., 2008) it is shown that the BTZ black hole with a scalar field included provides the framework for an interpolation between two different CFT-s, one at the infinite radius boundary and one at the outer Schwarzschild radius.[26]

---

[26]More on this subject is given in section 10.3 of chapter 10.

Chapter 4

Physical Preliminary: CFT the Theory at the End of 2-Dimensional RG

> Their souls rose free of all they'd been before;
> The past and all its actions were no more.
> Their life came from that close, insistent sun
> And in its vivid rays they shone as one.
> There in the Simorgh's radiant face they saw
> Themselves, the Simorgh of the world - with awe
> They gazed, and dared at last to comprehend
> They were the Simorgh and the journey's end.
> They see the Simorgh - at themselves they stare,
> And see a second Simorgh standing there;
> They look at both and see the two are one,
> That this is that, that this, the goal is won.
>
> Farid ud-Din Attar, *The Conference of the Birds*

In the book *The Conference of the Birds*, Farid ud-Din Attar describes the journey of a flock of birds to their god Simorgh. Though the story is actually an allegory of the spiritual journey of the sufi, it is also quite analogous to the journey of theories in the same universality class to their critical point by renormalisation flow.

Just as the birds, the theories start out very different from each other, but after a troublesome journey they end up as one and the same theory in the critical point, just as the birds discover that at the end of the journey, they are the Simorgh and they are one, completely washed of their individual traits, arrived at an eternal state of oneness. In 2-dimensional space, this state is described by a CFT.

In this chapter we will give a more formal introduction to this Renormalisation Group theory (RG), and how this faboulus theoretical construct helps us to identify quantum and statistical mechanics systems at criticality and to divide the space of all possible theories into universality classes, that is, classes of theories that flow to the same critical point under the RG-flow.[1]

After a description of RG we will briefly describe critical behaviour in statistical mechanics.

This chapter then gives a specialisation to the 2-dimensional case, where the critical points correspond to conformal field theories (CFTs). We will give a compilation of CFT for the purpose of this work. This compilation will hopefully show us why 2-dimensional theories are special.

4.1 Renormalisation Group Theory

Imagine the set of all conceivable QFTs as a topological space.[2] We see that each point/theory in this space can be described by the geometry of the spacetime on which it resides and by the couplings between the fields it contains.

We specialise initially to looking at subspaces consisting of theories on the same space-time geometry,[3] and see that they are in fact vector spaces over $\mathbb{R}$ where the directions are given by the different couplings, so that each point is parametrised by the values of its coupling constants.

In this space, we can choose one starting point and move through the space in the following manner:

Using the path integral formulation of QFT with some high-energy cutoff Λ,[4] integrate out the modes with the highest energy, and rewrite in terms of the lower energy modes. Then perform a scaling transformation s.t. the cutoff energy regains its value Λ. This process is called renormalisation.

This will in general change the values of the coupling constants, and the path traced through parameter space by repeatedly performing an infinitesimal version of this procedure is called the renormalisation flow.[5]

---

[1] The description given here is based on the treatment in (Peskin and Schroeder, 1995) and the beautiful description in (Lütken, 2006).

[2] We assume the reader to be familiar with the subject of QFT. For introductions to the subject the reader is refered to such sources as (Mandl and Shaw, 2007) and (Peskin and Schroeder, 1995) or any other of the now quite numerous introductions to this subject.

[3] We may also have flows in geometry, especially for compact geometries with length scales, changing the size of the space.

[4] Not having a cutoff is saying that you have an infinitely high resolution measuring device, which is highly unphysical.

[5] If the space has length scale, the RG-flow will in general also change the geometry of the spacetime.

In most cases we change position with RG-flows, but there are some initial points from which the RG-flow is stationary. These points are called RG fixed points.

Near the fixed points, theories may flow away from the fixed point, into it, or both, depending on the direction in which you move from the fixed point. Such fixed points are called repulsors/sources, attractors/sinks or saddle points, respectively.

All theories in the space, except repulsive fixed points, flow into attractor or saddle points. Theories that flow to the same fixed point are said to be in the same universality class.[6]

The direction and speed of the RG-flow at a given point, is given by its directional derivatives. The first derivatives are called the β functions. Near fixed points, the β functions[7] obey the so-called Callan-Symanzik equation.[8]

4.2 Statistical Mechanics and Critical Phenomena

Statistical mechanics is a wonderful framework. With a few assumptions about the interactions and energy levels of microscopic particles and an assumption of the kind of ensemble we are dealing with, we may calculate a suitable partition function and make predictions about emergent quantities like temperature and magnetisation for systems consisting of 10^{23} particles.

However, different assumptions about the microscopic properties of the system may have a large impact on the macroscopic theory. This yields a richness but also an unsatisfactory lack of order. Systems at criticality and the renormalisation flow discussed above are keys to ordering this picture of theories.

Phase transitions are *"characterized by a sudden and qualitative change in the macroscopic properties of the system as the temperature (or some other control parameter) is varied"* (Di Francesco et al., 1997). In the same book, (Di Francesco et al., 1997), we find the distinction between first order and continuous, or second order, transitions. *"First order transistions are characterized by a finite jump in the energy U"* [...] *"at the transition temperature"*. In continuous transitions there is no such latent heat, and no *"abrupt change in the average value of microscopic quantities"*. Instead it is in the derivatives of these quantities that we find the discontinuities.

In this work, we are mainly interested in the continuous phase transitions because of their relation to conformal invariance (Di Francesco et al., 1997). We will call these transitions *critical points* of the theory, and

[6] It is easy to show that being in the same universality class is an equivalence relation.

[7] One for each direction in theory space.

[8] See (Peskin and Schroeder, 1995) for more.

Exponent	Definition
α	$C \propto (T - T_c)^{-\alpha}$
β	$M \propto (T - T_c)^{\beta}$
γ	$\chi \propto (T - T_c)^{-\gamma}$
δ	$M \propto h^{1/\delta}$
ν	$\xi \propto (T - T_c)^{-\nu}$
η	$\Gamma(n) \propto \|n\|^{2-d-\eta}$

Table 4.1: Critical exponents and their definitions. C is the heat capacity. M is the total magnetization. χ is the susceptibility. h is the external field. ξ is the correlation length and $\Gamma(n)$ is the pair correlation function. The table is taken from (Di Francesco et al., 1997) and there you can also find the specific values of the exponents in the Ising model.

we notice that they exhibit a divergence in correlation length.[9].

One of the interesting features of systems at criticality is that different physical systems may exhibit similar behaviour around the critical point. Being physicists, we characterise the system at criticality by parameters which we can measure. Near criticality, we can for instance look at the degree of divergence when the temperature approaches its critical value. This is measured by the critical exponents. In fact, we shall see that we use this characterisation because the theories with equal critical behaviour are actually theories that flow to the same fixed points in RG-space, and that the critical exponents can be expressed in terms of the β functions at this point. Theories that flow to critical theories with identical critical exponents are said to be in the same universality class.

A table of the critical exponents and their names is shown in table 4.1.

4.2.1 Criticality and scale invariance

Since the correlation length ξ diverges at the critical point, the system at criticality has to have scale invariance. This can be understood through the following heuristic argument.

What does it mean that the correlation length diverges? Well, it means that the particles 'feel' other particles at an arbitrary distance. This yields a self-similar behaviour of the particles. This means that on one scale you see clusters of particles behaving in a correlated manner. Zooming out you see clusters of clusters behaving in much the same way as the particles at the scale below. Continuing outwards, the system always looks "the same". This leads us to conclude that the system is invariant under global scaling.

However, if we zoom out in one part of what we see, but not in another

[9] See (Di Francesco et al., 1997)

part, we still see that the system looks the same (except maybe on the boundary). This viewpoint leads us to conclude that the system is also invariant under local continuous scaling transformations, so the critical system is scale invariant.

4.2.2 RG-flows and the QFT- stat-mech duality at criticality

The fact that critical systems are scale invariant leads to a correspondence to QFT. Again, we justify this statement by a heuristic argument based on our knowledge of RG-flows.

In deriving the path integral formulation of a quantum field theory,[10] we start with a lattice of points through which the field can propagate. In the zero limit of lattice distances, we come to the quantum field theory. Upon renormalising the theory, we introduce a scale to the theory and then integrate out high-energy (short-distance) modes, yielding an effective action amongst the low-energy (long-distance) modes.

We saw above that at fixed points of this RG-flow, the theory is in fact scale invariant.[11]

We observe the same self-similar pattern as in the case of the critical statistical mechanics systems. The theory is blind to scaling. If we zoom out and look only at long-range interactions, the theory describing the system stays the same. Only a pertubation in the coupling constants can (possibly) move us away from the fixed point.

In the functional formalism it is easy to see that the thermodynamic limit of the lattice statistical mechanics model essentially is the same as the quantum field theory in the critical point/RG fixed point. The reason is that we saw that the quantum field theory emerged as the limit of a theory on a lattice, the limit where the lattice spacing goes to zero. When we look at the statistical mechanics system at criticality we have seen that the theory is blind to scaling. In the thermodynamic limit we can scale arbitrarily, even to the limit were the lattice spacing goes to zero. In this limit we see the quantum field theory emerging.

We see that for every statistical mechanics system at criticality, we can find a corresponding quantum field theory at an RG fixed point. Armed with the renormalisation group theory, we may now learn a lot about the statistical mechanics systems, or at least about their universality classes.

[10] For a more thourogh treatment of this subject, see for instance (Peskin and Schroeder, 1995) by which this section is largely inspired.

[11] In the description of RG-flows above, we saw that the RG-transformation is in fact a scaling transformation.

4.3 2-Dimensional Criticality and CFT

Going to two dimensions, the power of the renormalisation group theory becomes quite extraordinary. The reason for this is that scale invariance and conformal invariance are the same in two dimensions. The field theory we find at the critical point is then not only a quantum field theory but also a conformal field theory.

In the next chapter, we will look closer at the C-theorem, which is a theorem that puts strong restrictions on the possible RG-flows in two dimensions. Now we will, however, state and derive or at least justify some of the facts about conformal field theories that may be useful. All the stated facts can be found in (Di Francesco et al., 1997) and in (Ginsparg, 1990), upon which this section is heavily based. We will also make use of the geometric derivation and approach found in (Nash, 1991, fourth edition 2003). The latter approach is very useful in making conformal field theory understandable to the physicst familiar with normal, finite dimensional Lie group symmetries.

4.3.1 CFT symmetry group and algebra

We have observed that at the critical point the theory is invariant under so-called conformal transformations. Just as in gauge theory, we will now identify the symmetry group and use it to identify the fields with representations of the group algebra in the same way as done in (Nash, 1991, fourth edition 2003).

The conformal transformations are transformations that obey the conformal equations, that is, if we have a conformal transformation of the coordinates $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, then $\epsilon^\mu(x)$ satisfies (Di Francesco et al., 1997):

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \frac{2}{d}\partial_\rho \epsilon^\rho(x) g_{\mu\nu} \tag{4.1}$$

where d is the dimension of the space and $g_{\mu\nu}$ is the metric.

As shown in (Ginsparg, 1990), the conformal equations correspond to the Cauchy-Riemann equations in two dimensions. So the conformal transformations correspond to analytic or anti-analytic functions or linear combinations of the two. We will for obvious reasons not be interested in the constant maps.[12]

Thus our conformal group splits into two identical parts; holomorphic or anti-holomorphic transformations. Since the groups are identical and

[12]We mod out by the intersection between analytic and anti-analytic functions; the constant map. We will assume that these constant maps are not used in the remainder of this work.

do not intersect, we may look at only one of them. The other will yield an identical commuting algebra and subsequent fields.

Before identifying the group and its algebra, we observe that the group is in fact infinite. This is most readily seen if we use the polynomials $\{z^n\}, n \in \mathbb{Z}$ as a basis to span the group.[13] [14] This basis is clearly infinite.[15] The fact that the symmetry group is infinite is the source of the great power of conformal field theory in two dimensions. With an infinite symmetry group, the orbifold we get from modding out by the symmetry is exceedingly simple.

We have observed that the group we want to investigate is the group of holomorphic transformations. In (Nash, 1991, fourth edition 2003) it is observed that all such transformations are completely defined by their values on the unit circle so they are given by the group $\mathrm{Diff}(S^1)$ of diffeomorphisms[16] of S^1. The algebra of this group is the Virasoro algebra (Nash, 1991, fourth edition 2003):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{4.2}$$

where L_n, L_m are the generators of the algebra and c is called the central charge.

Such Virasoro algebras may occur in many situations where there is not conformal symmetry.[17] For instance, there is one residual Virasoro algebra left in the non-critical Ising model (Cardy, 1990).

The transformations given by the generators L_0, $L_{\pm 1}$ are special. These transformations are the global conformal transformations, i.e., they are bijections of the entire Riemann sphere. Together with their anti-holomorphic counterparts, L_{-1} generates translations, L_0 scale transformations and rotations, and L_1 generates special conformal transformations.[18]

To summarise, every conformal field theory is given by two such Virasoro algebras, Vir and $\overline{Vir}$, with the same central extention[19], $c = \bar{c}$. The Ising model at criticality, which we will come back to later, has $c = 1/2$.

[13] Locally conformal transformations need only be meromorphic.

[14] Remember that we omit the constant maps.

[15] Though the smallest of infinities since it is countable.

[16] A diffeomorphism is a homeomorphism that not only preserves continuity but also differentiability.

[17] In fact, by the Sugawara construction, they may be found whenever a Kac-Moody-algebra is present. See (Nash, 1991, fourth edition 2003) for more.

[18] See (Di Francesco et al., 1997).

[19] Formally, the group algebra need only have the first term in equation (4.2). However, without a central extention like the one given by the second term, the resulting theory is trivial. See (Nash, 1991, fourth edition 2003) for more on this subject.

4.3.2 Representations and field content

We now want to find nontrivial irreducible representations of this algebra. Such a representation with L_0 bounded from below is, rather counter-intuitively, called a highest-weight representation. The fact that $\mathrm{spec}(L_0)$ is bounded from below is natural, since in applications this is an energy (Nash, 1991, fourth edition 2003).

A highest-weight representation is specified by two numbers (c, h) where c is the central charge and h is the lowest L_0 eigenvalue.[20] The lowest eigenstate $|h\rangle$ is a 'vacuum state', and by assumption and use of the commutation relations given in the algebra:

$$L_0|h\rangle = h|h\rangle \qquad L_n|h\rangle = 0 \quad \forall \quad n \geq 0.$$

For the theory to be unitary, the values of c and h are restricted. Their possible values are (Friedan et al., 1984):

$$c \geq 1, \qquad h \geq 0 \tag{4.3}$$

or

$$\begin{array}{l} c = 1 - \frac{6}{(m+1)(m+2)} \\ \\ h = \frac{((m+2)p-(m+1)q)^2-1}{4(m+1)(m+2)} \end{array} \quad \left\{ \begin{array}{l} m = 1, 2, \ldots \\ p = 1, 2, \ldots, m \\ q = 1, 2, \ldots, p \end{array} \right. \tag{4.4}$$

In this work we will focus on the Ising model. At the critical point, the Ising model is, as already pointed out, described by a conformal field theory of central charge $c = 1/2$, the second of the discrete models found in equation (4.4).[21] Sometimes we will also show results for the Gaussian model which is a $c = 1$ model, i.e. one of the continuous models in equation (4.3).

Descendant fields - an eigenvector basis

The highest-weight states $|h\rangle$ are called primary fields. All theories contain a primary field with $h = 0$ called the identity field.

[20] In fact, we need three numbers to describe an entire state $(c, h, \overline{h})$. Though the central charge is the same for both $Vir, \overline{Vir}$, we may choose a highest-weight state that has different eigenvalues for L_0 and $\overline{L}_0$. A primary field should therefore be given as a field $\phi_{h,\overline{h}}(z)$ corresponding to a highest-weight state $|h, \overline{h}\rangle$, where $\overline{h}$ is not in general the complex conjugate of h. h and $\overline{h}$ are called the holomorphic and anti-holomorphic dimension of the primary fields to which they belong.

[21] The one with m = 2. The model with $m = 1$ is in fact a trivial model. A simple proof of this is given in (Nash, 1991, fourth edition 2003).

The theory, in general, contains many more fields than the primaries. However, the primary fields are cyclic and all other fields may be generated from them. Each primary field $|h\rangle$ generates a generally infinite coloumn of secondary fields by operation with the L_{-n}-generators: $L_{-1}^{\alpha_1} L_{-2}^{\alpha_2} \dots L_{-s}^{\alpha_s}|h\rangle$. Such fields are called descendant fields of $|h\rangle$.

Each application of an L_{-n} operator raises the L_0 eigenvalue of the resulting field by n. If a descendant of $|h\rangle$ has the L_0 eigenvalue $N + h$, we say that it is a descendant at level N. The number of descendants of a field $|h\rangle$ at level N equals the partition of N. The energy momentum tensor $T_{\mu\nu}$ has three independent components. They are all level 2 descendants of the identity field.

The primary fields and their descendants form an eigenvector basis for the space of fields. In principle, all fields in the conformal field theory may be written as a linear combination of primary fields and their descendants. The subspace spanned by the descendandants of a single primary field $|h\rangle$ is called a *Verma module* and will be denoted by $\mathcal{V}_h$.[22]

The Verma modules together also form an algebra which determines how fields from different modules may be fused together.

Sometimes the eigenvector basis formed by a Verma module may be reducible. For instance, there may be a singular vector $|\chi\rangle$ at some level in the Verma module, that is, a vector that is annihilated by all L_n such that $n > 0$. Then $|\chi\rangle$ and all its descendants are orthogonal to the whole Verma module.[23] We can reduce the Verma module by quotienting out all these singular submodules.

Minimal models

The Verma modules in the discrete models given in equation (4.4) are all reducible. This means that their matter content is finite. It is then a finite task to find all primary fields from equation (4.4) and all descendants. Using the algebra, we can then find all the interaction properties.[24]

For instance, we can determine the so-called fusion coefficients $\mathcal{N}_{ij}^k$, that determine the fusion algebra between the fields of different modules. The fusion of the field ϕ_i and ϕ_j will then be given as:

$$\phi_i \times \phi_k = \sum_k \mathcal{N}_{ij}^k \phi_k \tag{4.5}$$

Whence $\mathcal{N}_{ij}^k$ is symmetric in the lower indices.

For the Ising model, the content is then so small that this is the only such theory. However, at larger m-value levels of equation (4.4), there may

[22] $\mathcal{V}_h$ includes $|h\rangle$ itself since $|h\rangle = L_{-q}^0|h\rangle$. Hence $|h\rangle$ is a level 0 descendant of itself.

[23] This can be seen from the algebra. The proof is written out in (Di Francesco et al., 1997).

[24] Though this may be tedious, the fact that it is finite makes it doable.

well be subspaces of fields that are closed under fusion. At that value of c there are different theories; the full theory, and theories spanned by the subspace fields.

4.3.3 Transformation properties

Primary fields have simple transformation properties under conformal transformations (Di Francesco et al., 1997).

Let a conformal map be given by $z \to w(z)$, $\overline{z} \to \overline{w}(\overline{z})$. We will also now denote the primary field of conformal dimensions $(h, \overline{h})$ by $\phi(z, \overline{z})$:

$$\phi'(w, \overline{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}} \phi(z, \overline{z}) \tag{4.6}$$

Correlation functions of primary fields transform simply by multiplying the old correlation function by the transformation factors for each primary field.

Secondary fields have more complicated transformation properties. However they transform just as simply under global conformal transformations, though their dimensions are given by $(N + h, \overline{N} + \overline{h})$.

In general, a secondary field $A(z, \overline{z})$ of conformal dimensions $(N + h, \overline{N} + \overline{h})$ transforms as:

$$A(z, \overline{z}) \to \left(\frac{dw}{dz}\right)^{-(h+N)} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-(\overline{h}+\overline{N})} A(z, \overline{z}) + \text{extra} \quad \text{terms} \tag{4.7}$$

We will see an explicit form of such a transformation in section 7.1 of chapter 7.

4.3.4 Modular invariance - consistent theories on the torus

In CFTs on the plane, the holomorphic and anti-holomorphic sectors are completely independent. The two sectors are often called left and right movers, and since they are independent, any combination of left and right Verma modules is in principle possible.

However, the plane is only a very special 2-dimensional geometry. It represents a sort of critical point in theory space and we can imagine perturbing the geometry away from it. This should lead to constraints on the combinations of Verma modules that are possible, i.e., taking a slightly different geometry and slowly deforming it to a plane, should not change the combination of Verma modules present (Di Francesco et al., 1997).

Armed with the classification theorem of 2-dimensional geometry, this should not be too difficult. Since the sphere is just a one point compactification of the plane, it should introduce no new constraints. Also, the cylinder mapping given in equation (3.1) is conformal, so the cylinder should not impose constraints.[25] But as we have seen in section 3.2.3, consistency on the torus imposes the constraints of modular invariance.

In general, this puts rather strong constraints on the partition function of the theory.[26] Defining $q = e^{2\pi i \tau}$, and the character of a primary field χ_h

$$\chi_h(q) = \mathrm{Tr}_{\nu_h} q^{L_0 - c/24} = \sum_N d_h(N) q^{h-c/24+N} \tag{4.8}$$

where $d_H(N)$ is the number of descendant states at level N, the partition function can be written:

$$Z(\tau, \overline{\tau}) = \sum_{h,\overline{h}} n_{h,\overline{h}} \chi_h(q) \chi_{\overline{h}}(\overline{q}) \tag{4.9}$$

The possible values of $n_{h,\overline{h}}$ is constrained by modular invariance. One possible modular invariant Z is always the diagonal sum, $n_{h,\overline{h}} = \delta_{h,\overline{h}}$:

$$Z(\tau, \overline{\tau}) = \sum_h \chi_h(q) \chi_h(\overline{q}) \tag{4.10}$$

but there may also be others. All this may be found in (Cardy, 2008).

Two Verma modules are said to fuse into one other Verma module. This is given by a fusion algebra. If one can find a subset of modules closed under fusion, it may be possible to form a modular invariant partition function from the diagonal sum of characters of these submodules.[27] In the case of a theory of conformal charge $c = 1/2$, the constraint of modular invariance fully defines the theory. We will see this in chapter 6.

4.3.5 Geometries with boundaries - BCFT

It is very natural to look at theories with a boundary in addition to those without. A geometry with boundary breaks the conformal invariance, but this does not mean that conformal field theory has no part in their description. There is in fact a theory called boundary conformal field theory (BCFT) that enables us to use the power of CFT to describe it. We will give a brief review of this theory here. We will follow the material given in (Cardy, 2008) quite closely, however the text is also inspired by the treatment given in (Di Francesco et al., 1997).

[25] In general, we will not look at constraints posed by geometries with boundaries, since the object is then locally different from the plane (near the boundary). We will also in general only consider orientable, compact surfaces. (Non-orientability seems a little unphysical and one-point compactification takes care of non-compactness).

[26] All measurable quantities can be expressed in terms of the partition function. Hence constraints on the theory may be imposed by demanding that the partition function obey them.

[27] The description of modular invariance given here can be found in (Cardy, 2008). A more thorough treatment can be found in (Di Francesco et al., 1997).

The simplest boundary surface is that corresponding to the plane, that is, the disk, and by analogy with our previous use of complex coordinates, we will assume it to be the Poincaré disk. For those familiar with hyperbolic geometry and the Poincaré disk, recall that the disk and the upper half plane are the same thing,[28] taking the boundary of the disk to the real axis (one point compactified with the point at infinity) and the interior of the disk to the upper half plane.[29] We will here make use of both the upper half plane and the Poincaré disk, interchanging the two concepts as we see fit.

For a conformal transformation of a surface with boundary to make sense it needs to preserve the boundary and the boundary conditions. For the upper half plane, this means it needs to preserve the real axis. This couples the holomorphic and anti-holomorphic sectors, and in fact only half the conformal genrators, i.e. only one Virasoro algebra, remains:

$$Vir \times \overline{Vir} \rightarrow Vir.$$

.

Before we continue, we should define reasonable boundary conditions. Not all boundary conditions are allowed. A sensible choice would be to demand that no momentum flows across the boundary. This is called a conformal boundary condition.[30] However, this boundary condition does not uniquely determine the possible states at the boundary.

We saw in the case of the normal CFT, that the theory was strongly constrained by the theory on the torus, and we will now employ the same method to investigate BCFT. The boundary surface corresponding to the torus is the annulus.

How does our boundary setting give us an annulus? Imagine that the boundary condition is not the same along the whole boundary. In the disk setting we could just think of employing two different conditions a and b. However, instead of having only these two, we separate the domains of these boundary conditions by an area of free boundary conditions. Different pictures of what this looks like are shown i figure 4.1.

[28]For a review of this, see (Jahren, 2007b).

[29]A simple way to understand that the Poincaré disk and the upper half plane are the same is by considering stereographic projection. Insert a plane along the real axis and peripendicular to the imaginary axis of the plane in the picture of the steoregraphic projection 3.2. This plane will cut the Riemann sphere in two. The hemisphere above this plane (positive y-component) is taken into the upper half plane by stereographic projection, the boundary circle is taken to the compactified real line. Hence a hemisphere is identified with the upper halfplane. But it is easily seen that a hemispehere and a disk are the same. Actually the Poincaré disk is made by rotating this hemisphere to the northern hemisphere, then taking each point to the point directely underneath it on the plane.

[30](Cardy, 2008) argues that all boundary conditions will flow to this condition with the RG-flow. Hence the critical state for a geometry with boundary leads to this boundary condition.

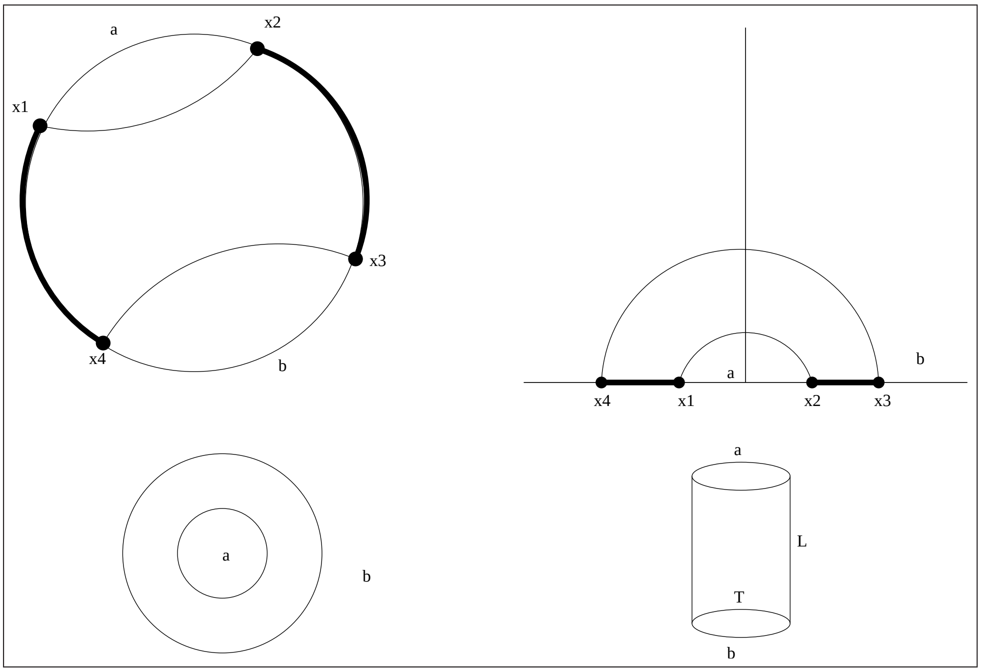

Figure 4.1: This figure shows different representations of the annulus. On the top left the disk has four segments of different boundary conditions. The segments with the thick line have free boundary conditions, we may consider them to be identified. The other two segments have boundary conditions a and b respectivly. Geodesics between the identified free boundary condition segments are also drawn. On the top right, this picture has simply been transported to the upper half plane. On the left below, we see the picture transported to a radially quantized ξ-plane. Here the boundary condition a is the initial state of the system, b is the final state. On the lower right, the annulus is shown as a cylinder of finite length L and circumference T. An appropriate picture may be a spin chain of lenght L with boundary conditions a and b at the two ends. This system is then propagated for a time T and has periodic boundary conditions in the time direction.

As shown in figure 4.1 the segments of free boundary conditions may well be identified, forming a cylinder of finite length with different boundary conditions on its edges. We can then have two situations; time can flow along the cylinder or around the cylinder. In the first scheme, the Hamiltonian depends on the boundary conditions, we will denote it H_{ab}. In the second, the system propagates from one boundary state to the other.

Let us start with a cylinder of temporal circumference T and spatial length L.[31] We are now in the first scheme, and the partition function is given as:

$$Z_{ab}(q) = \mathrm{Tr} e^{-(\pi T/L)H_{ab}} = \mathrm{Tr} q^{H_{ab}} \tag{4.11}$$

where $q = \exp(2\pi i\tau)$ as in the torus case, but the modular parameter is $\tau = iT/2L$.

We have already assumed the system to be conformal, and as argued above there is a Virasoro algebra equal to the normal CFT one. That is, all the states are in their irreducible representations, which are just the old Verma modules. If n^h_{ab} is the number of copies of the representation h, the partition function may be written as:

$$Z_{ab}(q) = \sum_h n^h_{ab}\chi_h(q) \tag{4.12}$$

where $\chi_h(q)$ is the character of the representation h as given by equation (4.8).

In minimal models the characters transform as follows under the transformation $\mathcal{S}$:

$$\chi_h(q) = \sum_{h'} S_{hh'}\chi_{h'}(\tilde{q}) \qquad \tilde{q} = e^{-2\pi i/\tau} = q(\mathcal{S}(\tau)) \tag{4.13}$$

where $S_{hh'}$ is called the modular matrix.

This transformation now interchanges L and T.[32] This interchange takes us from the first to the second scheme, so the partition function is a trace of the Hamiltonian that generates translations along σ, the cylinder height ($w = t + i\sigma$ the cylinder coordinate) given the initial and finite states $|a\rangle$ and $|b\rangle$.

To find this we map the cylinder into the plane via the transformation:

$$\xi = e^{-2\pi i(t+i\sigma)/T} \qquad \text{or} \qquad w = i\frac{T}{2\pi}\ln\xi \tag{4.14}$$

Here, the cylinder height maps to the radial coordinate, and we get a radially quantized plane. The two boundaries are consentric circles where

[31] See the lower right part of figure 4.1.

[32] More precisely, there is a factor of four as well. If $\delta = T/L$, this transformation takes $\delta \to -4/\delta$.

the boundary conditions are imposed. This is shown in the lower left part of figure 4.1.

Now, let L_n^ξ, $\overline{L}_n^\xi$, be the Virasoro generators on the ξ-plane. Then the Hamiltionian is:

$$\tilde{H} = \frac{2\pi}{T}(L_0^\xi + \overline{L}_0^\xi - \frac{c}{12}) \tag{4.15}$$

Thus, the partition function is given as:

$$Z_{ab}(q) = \langle a|e^{L\tilde{H}}|b\rangle = \langle a|(\tilde{q}^{1/2})^{L_0^\xi + \overline{L}_0^\xi - \frac{c}{12}}|b\rangle \tag{4.16}$$

Our conformal boundary conditions leads to the condition that:

$$L_n^\xi|\alpha\rangle = \overline{L}_{-n}^\xi|\alpha\rangle \qquad \text{where} \qquad |\alpha\rangle \in \{|a\rangle, |b\rangle\} \tag{4.17}$$

In particular $h = \overline{h}$ for the allowed boundary states.

Inserting a complete basis and comparing with the result in the first scheme, we get the so-called Cardy conditions:[33]

$$\begin{aligned} n_{ab}^h &= \sum_{h'} S_{hh'}\langle a|h'\rangle\langle h'|b\rangle \\ \langle a|h'\rangle\langle h'|b\rangle &= \sum_h S_{h'h} n_{ab}^i \end{aligned} \tag{4.18}$$

In the minimal case,[34] unitarity leads to positivity of S_{0h}, and we can use this to define boundary states, one corresponding to each representation:

$$|\tilde{0}\rangle = \sum_h S_{0h}^{1/2}|h\rangle \tag{4.19}$$

$$|\tilde{h}\rangle = \sum_{h'} \frac{S_{hh'}}{\sqrt{S_{0h'}}}|h'\rangle \tag{4.20}$$

Inserting this into the second Cardy condition with boundary states $|h'\rangle$ and $|h''\rangle$ yields:

$$\sum_i S_{ih} n_{\tilde{h}'\tilde{h}''}^i = \frac{S_{h'h}S_{h''h}}{S_{0h}} \tag{4.21}$$

This equation is, however, identical to the Verlinde formula[35] with $n_{h'h''}^i$ exchanged with the fusion coeffisients $\mathcal{N}_{h'h''}^i$. The fusion coefficients are known in each model, and hence we can completely solve the boundary system and interpret the boundary states and their fusion.[36]

[33] The two conditions are in fact equivalent.

[34] And in general for diagonal CFTs.

[35] The Verlinde formula gives the relation between the fusion coefficients $\mathcal{N}_{h'h''}^i$ and the elements of the modular matix S_{jk}. For more on this see (Di Francesco et al., 1997).

[36] The same fusion rules applies for the $|\tilde{h}\rangle$ states as for the $|h\rangle$ states.

Chapter 5

Zamolodchikov's C-theorem

> Humpty Dumpty sat on a wall,
> Humpty Dumpty had a great fall.
> All the king's horses,
> And all the king's men,
> Couldn't put Humpty together again.

English nursery rhyme

Humpty Dumpty may seem only to be a silly nursery rhyme stating obvious facts about eggs,[1] but we can also use it as an analouge for theories in RG space. Starting as a theory at a repulsive or saddle fixed point, Humpty Dumpty sits comfortably on the wall. He is then perturbed out of this fixed state, causing him to fall or flow along an RG flowline towards a new attractive or saddle fixed point of lying on the floor in innumerable pieces.

So far so good. But is it inherently true that Humpty Dumpty cannot be put together again? That is, is there no RG flow line that can take the theory back to his state on the wall, or possibly a state in the falling flow line? That such flows are not possible may seem natural, but it is in fact not true in general. In this chapter, we will explore the famous Zamolodchikov's C-theorem that proves this in the case of the 2-dimensional plane. We will also explore an alternative proof, review the search for generalisations, and find that in fact it is not generally true.

5.1 Zamolodchikov's C-theorem

As we have seen, the renormalisation group is a profound concept. Zamolodchikov's C-theorem gives us detailed information about the

[1] Or historical facts about cannons. See for instance the Wikipedia page on Humpty Dumpty.

possible flows in two dimensions. It also gives us a prescription for finding the C-function for a specific field theory. We will now state the theorem as it was first stated by Zamolodchikov in (Zamolodchikov, 1986). The theorem is restated word for word, but where Zamolodchikov gave the equation for the Virasoro algebra, we simply refer the reader to (4.2) which is identical when the central charge c is denoted $\tilde{c}$. Zamolodchikov's own references have also been included.[2]

Zamolodchikov's C-theorem 1 *1. There exists a function $c(g) \geq 0$ of such a nature that we have*

$$\frac{d}{dt}c \equiv \beta^i(g)\frac{\delta}{\delta g^i}c(g) \leq 0 \tag{5.1}$$

(a repeated index implies summation). The equality in (5.1) is reached only at fixed points of the renormalisation group, i.e., at $g = g_*$ [$\beta^i(g_*) = 0$]

2. The fixed points (here and below we mean "critical" fixed points, at which the correlation radius is infinite (Wilson and Kogut, 1975)) are stationary for $c(g)$; i.e. we have $\beta^i(g) = 0 \rightarrow \delta c/\delta g^i = 0$. At the critical fixed points, the 2D field theory has an infinte conformal symmetry (Belavin et al., 1984). The corresponding generators $L_n, n = 0, \pm 1, \pm 2, \cdots$ form a Virasoro algebra as given in equation (4.2), where the numerical parameter $\tilde{c}$ (the "central charge") is an important characteristic of a conformal field theory (Belavin et al., 1984), (Friedan et al., 1984). It generally takes on different values for different fixed points; i.e. $\tilde{c} = \tilde{c}(g_*)$.

3. The value of $c(g)$ at the fixed point g_* is the same as the corresponding central charge in (4.2) i.e. $c(g_*) = \tilde{c}(g_*)$

5.2 Zamolodchikov's Proof of the C-theorem

The basic idea in proving the C-theorem lies in finding the C-function (or a candidate C-function) and then proving that it has the required properties.

I will now give a more detailed version of Zamolodchikov's own proof which may be found in (Zamolodchikov, 1986), but the proof given here is written in the formalism of (Cardy, 1990) and is also inspired by the proof of the C-theorem given in section 6.2 of (Cardy, 1990). The reason for giving this proof is twofold. Firstly, we would like to have a proof where all the details are worked out. Secondly, the proof prescribes a way of calculating the actual C-function, and we will make use of this prescription in chapter 6.

The proof starts by assuming translational and rotational symmetry of the field theory. This assumption is quite natural in most cases and can be reasoned as follows.

[2]All of these references are classics in the developement of CFT and should be read in their own right.

Translational invariance is an assumption of the homogenity of space, and is rooted in the arbitraryness in choosing orgin.[3] However, this assumption might not be entirely correct on a statistical mechanical 2-dimensional lattice. It will be in the limit of infinitely large systems, and we now assume that it is here we find our effective field theory.

Similarly, rotational symmetry originates in the arbitrary choice of axis. This may not be true for all crystal lattice structures (where the couplings may dependen on direction). However, this may be solved by a suitable change of coordinates.[4]

The spatial symmetries we have assumed above lead to the existence of a local energy momentum tensor $T_{\mu\nu}$ (by Noether). By rotational symmetry it is symmetric $T_{\mu\nu} = T_{\nu\mu}$, and it is conserved $\delta^{\mu} T_{\mu\nu} = 0$.

In complex coordinates we write the three independent components as:

$$T = T_{zz}, \qquad \overline{T} = T_{\overline{z}\overline{z}},$$

$$\Theta = 4T_{z\overline{z}} = 4T_{\overline{z}z} = 2g_{z\overline{z}}T^{z}_{z} + 2g_{z\overline{z}}T^{\overline{z}}_{\overline{z}} = T^{z}_{z} + T^{\overline{z}}_{\overline{z}} = T^{\mu}_{\mu}.$$

So Θ is the trace of the energy momentum tensor. More on the metric used for the transitions above can be found in appendix II.

Define the following functions:

$$F(g, z, \overline{z}) = z^4 \langle T(z, \overline{z}) T(0, 0) \rangle \tag{5.2}$$

$$G(g, z, \overline{z}) = z^3 \overline{z} \langle T(z, \overline{z}) \Theta(0, 0) \rangle \tag{5.3}$$

$$H(g, z, \overline{z}) = z^2 \overline{z}^2 \langle \Theta(z, \overline{z}) \Theta(0, 0) \rangle \tag{5.4}$$

Because of reflection positivity, all of these functions are always positive. By translational and rotational invariance, correlation functions only depend on the distance between the points, so F, G and H only depend on the product $z\overline{z}$. The scaling properties of the theory will be given by the scaling of this distance given by a parameter m, where we put $z\overline{z} \to mz\overline{z}$.

From these we define our trial C-function

$$C(g, z, \overline{z}) = 2F - G - \frac{3}{8}H \tag{5.5}$$

Does this function meet the demands of the theorem? Before we start the proof, let us derive some properties that will be needed.

Conservation of the energy momentum tensor implies that:

---

[3] We almost always assume this without blinking. (Your lab is in principle just as good as mine for measuring the value of α.)

[4] See for instance (Cardy, 1990).

$$\partial^{\overline{z}} T_{z\overline{z}} + \partial^{z} T_{zz} = 0, \qquad \partial^{\overline{z}} T_{\overline{z}\overline{z}} + \partial^{z} T_{z\overline{z}} = 0,$$

and using the metric derived in appendix II yields:

$$\partial_{\overline{z}} T + \frac{1}{4}\partial_z \Theta = 0 \tag{5.6}$$

We want to take the correlation functions of (5.6) with $T(0,0)$ and $\Theta(0,0)$ to obtain relations between the derivatives of F, G and H.

Defining

$$\dot{} \equiv z\overline{z}\frac{\partial}{\partial(z\overline{z})} \tag{5.7}$$

When we let $z\overline{z} \to mz\overline{z}$, we see that $\dot{} \to \frac{d}{dm}$. So the derivative along the flow is given by $\dot{}$.

Since the functions F, G and H depend only on $z\overline{z}$, we want to express δ_μ in terms of the derivative operator (5.7).

What we get is

$$\partial_z = \frac{\partial z\overline{z}}{\partial z}\frac{\partial}{\partial z\overline{z}} = \frac{1}{z}\dot{} \qquad \partial_{\overline{z}} = \frac{\partial z\overline{z}}{\partial \overline{z}}\frac{\partial}{\partial z\overline{z}} = \frac{1}{\overline{z}}\dot{}$$

In the conservation of the correlation functions we need $\partial_{\overline{z}}F$, $\partial_{\overline{z}}G$, $\partial_z G$ and $\partial_z H$. These are:

$$\begin{aligned}
\partial_{\overline{z}} F &= \frac{1}{\overline{z}}\dot{F} = z^4 \partial_{\overline{z}} \langle T(z,\overline{z}) T(0,0)\rangle \\
\partial_{\overline{z}} G &= \frac{1}{\overline{z}}\dot{G} = \frac{1}{\overline{z}} G + z^3\overline{z}\partial_{\overline{z}} \langle T(z,\overline{z}) \Theta(0,0)\rangle \\
\partial_z G &= \frac{1}{z}\dot{G} = \frac{3}{z} G + z^3\overline{z}\partial z \langle T(z,\overline{z}) \Theta(0,0)\rangle \\
\partial_z H &= \frac{1}{z}\dot{H} = \frac{2}{z} H + z^2\overline{z}^2\partial z \langle \Theta(z,\overline{z}) \Theta(0,0)\rangle
\end{aligned} \tag{5.8}$$

Plugging in the result in (5.8) and multiplying with suitable factors of z and $\overline{z}$, we end up with the following equations:

$$\dot{F} + \frac{1}{4}(\dot{G} - 3G) = 0 \tag{5.9}$$

$$\dot{G} - G + \frac{1}{4}(\dot{H} - 2H) = 0 \tag{5.10}$$

Using these equations to eliminate G and express $\dot{G}$ in terms of $\dot{F}$, $\dot{H}$ and H we get

$$\dot{G} = 2\dot{F} - \frac{3}{8}(\dot{H} - 2H) \tag{5.11}$$

Now we are almost ready to prove the C-theorem. All we need is some conformal field theory results. The first is that in scale invariant theories, the energy momentum tensor is traceless. We will also use that scale invariance and conformal invariance are the same in two dimensions, and that a conformal field theory is generated by operators L_n that follow a Virasoro algebra as given in (4.2).[5] The central charge, $\tilde{c}$, of this CFT is defined by [6]

$$\langle T(z,z)T(0,0)\rangle = \frac{\tilde{c}}{2z^4} \tag{5.12}$$

Let us start with the positivity. We know that at all critical points $H = G = 0$ and $F = c/2 \geq 0$. We can move between theories, visiting all points in parameter space, if we start from repulsive critical points perturbing them in a suitable way, finally ending up at a different critical point. If the C-function is monotonous in the flow direction, we know that there cannot have been a minimum between the critical points, and so the C-function must have been positive all the time. But proving this is essentially the same as proving the rest of the first statement of the C-theorem.

Taking $dC/dm = \dot{C}$ and substituting equation (5.11) for $\dot{G}$ we find that

$$\frac{dC}{dm} = -\frac{3}{4}H \tag{5.13}$$

which is always negative (and hence C is monotone), by the positivity of H. The $\dot{C} \leq 0$ equality is reached when $H = 0$, i.e. at the critical points.

From the equality of scale and conformal invariance in two dimensions, it follows that the critical points are conformal points. Hence all of point (2) of the theorem 5.1 follows from CFT. By looking at the critical value of C and the definition of the central charge, we see that point (3) of the theorem also follows. This completes the algebraic proof of Zamolodchikov's C-theorem.

5.3 Friedan's Proof of the C-theorem

Although Zamolodchikov's theorem is perfectly rigorous, it is slightly dissatisfying in the sense that it may be difficult to understand the physical origin of the exact C-function.

Dan Friedan made a different proof based on spectral decomposition (Friedan, 1990), which may give us more insight. The proof presented in this section is based on (Cappelli et al., 1991), as well as a clarifying email from Friedan to Latorre (Friedan, 1990/1991?).

[5] The derivation for the anti-holomorphic part is identical, just replace all occurances of T with $\overline{T}$ etc.

[6] See for instance (Cardy, 1990).

The proof basically goes like this.[7] How can we construct a C-function that obeys the criteria of Zamolodchikov's C-theorem using unitarity and the conservation of the energy-momentum tensor? Thinking about what the C-function does may be a useful starting point.

The C-function is a decreasing function along the RG flow lines. As we move along the RG-flow, we integrate out high-momentum degrees of freedom. One reason we want a C-theorem, is that we want to quantify the loss of degrees of freedom, and we want to verify our naive expectation that this process is monotonous and irreversible. Usually there will not be new degrees of freedom suddenly as we integrate out old ones. In a sense, we want the C-function to measure the (density of) degrees of freedom in our theory.

The spectral function,[8] or at least the integral of it, does this job for a given system, so we try to construct our C-function from the spectral density.

In addition, we want to use the assumptions of the conservation of the energy-momentum tensor and the unitarity in our theory to prove that this trial function has the qualities we seek.

What we do is start with the correlator of the energy-momentum tensor.

$$\langle T_{ij}(x)T_{kl}(0)\rangle$$

We then rewrite this correlator by inserting the identity in terms of the integral of projectors onto mass-eigenstates.

$$I = \int_0^\infty \mathcal{P}_{\mu^2} = \int d\mu^2 \delta_+(\mathbb{P}^2 + \mu^2) \tag{5.14}$$

We can simplify the C-function by extracting the tensor structure of the resulting expression for the correlator. As the Fourier transformed expression allows only for Lorentz invariant tensor structures built from p_α and $g_{\alpha\beta}$, we need only consider linear combinations of 4-index objects built out of these tensors. The distinct objects of this type are $p_ip_jp_kp_l$, $p^2g_{ij}p_kp_l$, $p^2g_{kl}p_ip_j$ and $p^4g_{ij}g_{kl}$. For dimensional reasons they should be divided by $p^2+\mu^2$, so the correlation function must be of the form:

$$\begin{aligned}\langle T_{ij}(x)T_{kl}(0)\rangle \quad = \quad & \int_0^\infty d\mu c(\mu) \int \frac{d^2p}{(2\pi)^2} e^{ip_nx^n} \times \\ & \frac{(Ap_ip_jp_kp_l + Bp^2g_{ij}p_kp_l + Cp^2g_{kl}p_ip_j + Dp^4g_{ij}g_{kl})}{p^2+\mu^2}\end{aligned} \tag{5.15}$$

[7] Since our version of Zamolodchikov's proof is rigorous, we will not be as rigorous in the presentation of Friedan's proof, but rather focus on the physical insights that may be gained from his approach.

[8] For more on the spectral function, see for instance (Peskin and Schroeder, 1995).

We now apply conservation of the energy momentum tensor $\partial^\mu T_{\mu\nu} = 0$ to the correlator, where

$$\partial^\mu = g^{\mu\alpha}\partial_\alpha = g^{\mu\alpha}\frac{\partial}{\partial x^\alpha}.$$

For instance

$$\begin{aligned}
\partial^i\langle T_{ij}(x)T_{kl}(0)\rangle &= \int_0^\infty d\mu c(\mu)\int\frac{d^2p}{(2\pi)^2}e^{ip_nx^n}ip_m g^{im}\times \\
&\quad \frac{(Ap_ip_jp_kp_l + Bp^2g_{ij}p_kp_l + Cp^2g_{kl}p_ip_j + Dp^4g_{ij}g_{kl})}{p^2+\mu^2} \\
&= \int_0^\infty d\mu c(\mu)\int\frac{d^2p}{(2\pi)^2}e^{ip_nx^n}\times \\
&\quad \frac{(A+B)p^2p_jp_kp_l + (C+D)p^4p_jg_{kl})}{p^2+\mu^2} \\
&= 0, \qquad (5.16)
\end{aligned}$$

which leads us to conclude that $A = -B$, $C = -D$, thus eliminating B and C. Similarly an application of ∂^k leads us to $A = D$. The tensor part then takes the form

$$A\frac{(p_ip_j - p^2g_{ij})(p_kp_l - p^2g_{kl})}{p^2+\mu^2} \qquad (5.17)$$

There is an arbitrariness in the choice of A.[9] The choice of Friedan is $A = \pi/3$ (Cappelli et al., 1991).

Now the unitarity of the theory tells us that the correlator must be positive, but since $(p_ip_j - p^2g_{ij}) \geq 0$, this leads to the conclusion that $c(\mu) \geq 0$.

Dimensional analysis tells us that $c(\mu)$ must have dimension μ^{-1}. Thus, we must have $c(\mu) = c_0\delta(\mu) + K\mu^{-1}$. In the massless limit the second part will give rise to a divergence, and we are left with only the δ function in the scale-invariant/massless limit.

We now analyse the short and long distance behaviour of the correlators using complex coordinates $z = x + iy$ and the previous notation for the components of the energy-momentum tensor:

$$\lim_{z\to 0}\langle T(z)T(0)\rangle \to \frac{1}{2z^4}\int_0^\infty d\mu c(\mu) \qquad (5.18)$$

$$\lim_{z\to\infty}\langle T(z)T(0)\rangle \to \frac{1}{2z^4}\lim_{\epsilon\to 0}\int_0^\epsilon d\mu c(\mu) \qquad (5.19)$$

[9]This is actually just a normalization and we may adjust it to obtain the correct proportionality to the central charge at the critical point

We see that $c(\mu) = c_{IR}\delta(\mu) + c_1(\mu)$, and from our knowledge of the TT-correlator, we see that c_{IR} has the value of the central charge at the long distance fixed point in RG.

The short distance fixed point value is then:

$$c_{UV} = \int_0^\infty d\mu c(\mu) = c_{IR} + \int_0^\infty d\mu c_{smooth}(\mu) \tag{5.20}$$

Again, unitarity leads to the positivity of $\int_0^\infty d\mu c_1(\mu)$, and

$$c_{UV} \geq c_{IR}$$

Now we have seen that we can build a function with the right short and long distance fixed point values from $\int d\mu c(\mu)$. However, we need to impose the desired monotonic behaviour throughout the whole space of coupling constants. We can do this by introducing a smearing function $f(\mu)$ with the properties:

$$f > 0 \qquad f(0) = 1 \qquad \mu\frac{df}{d\mu} \leq 0$$

We want f to decrease exponentially as $\mu \to \infty$. Applying these criteria we define:

$$c(g(\Lambda)) = \int d\mu c(\mu) f(\mu) = \int d\mu c_1(\mu, \Lambda) f(\mu) + c_{LD} \tag{5.21}$$

By use of the Callan-Symanzik equation we find that

$$-\beta^i \frac{\partial}{\partial g^i} c = \Lambda \frac{\partial}{\partial \Lambda} c = \int d\mu c_1(\mu, \Lambda) \mu \frac{df}{d\mu} \leq 0 \tag{5.22}$$

In order to match this to the Zamolodchikov value we want

$$-\beta^i \frac{\partial}{\partial g^i} c = -6\pi^2 \beta^i \beta^j G_{ij},$$

where G_{ij} is the metric in the space of coupling constants around the fixed points.

Earlier in the article, (Cappelli et al., 1991) find that near a fixed point $\Theta = 2\pi\beta^i\phi_i$ and $G_{ij} = \langle\phi_i(x)\phi_j(0)\rangle = G_{ji}$, where ϕ^i are the relevant operators.

Using a new smearing function $h > 0$ which is exponentially decreasing for $\mu \to \infty$ ($h(0) = \text{constant}$) we can write

$$G_{ij} = \frac{2}{\pi} \int d\mu \rho_{ij} h(\mu). \tag{5.23}$$

We find that we obtain the correct C-function by matching h and f such that $\mu \frac{df}{d\mu} = -\mu^4 h(\mu)$. A careful consideration reveals that near fixed points, the linear version of the C-function must be of the form given by Zamolodchikov.[10] However, further away from the fixed point, our smearing functions may give us a different C-function than Zamolodchikov's.

This reflects a certain arbitraryness in the flow away from the fixed points, it may flow through the points on the flow line faster or slower depending on the integration scheme. This speed is unphysical, as it corresponds to the arbitraryness in choice of renormalisation scheme, familiar from ordinary QFT.

5.4 Consequences of the C-theorem

It is now in order to consider the consequences of the C-theorem.

Let us start off with the first part of the theorem. The mere existence of a positive function monotonically decreasing along the RG-flow lines is quite important in its own right. This existence means that we cannot have RG-loops in the 2-dimensional case.[11]

The proof of Friedan gave us some insight into why the existence of such a function is not unnatural, and what the C-function actually is. The C-function is a function that in some sense counts the degree of freedom of the theory. The monotonic decrease of the C-function means that the (density of) degrees of freedom decreases as we integrate high-momentum degrees of freedom out of our theory. The RG-flow in two dimensions is intrinsically irreversible.

The second and third parts of the C-theorem connects the C-function to the appropriate conformal field theory at the RG fixed points. This also gives us insight into how the space of theories is built. Starting with a minimal conformal model (for which $c < 1$), we cannot flow to a conformal theory with $c > 1$. More generally we see that a vast number of configurations of the RG phase space are not allowed. For example, the

[10] To see this, consider $h(\mu)$ and its relation to the correlation function H that we found in the section on Zamolodchikov's proof. Not considering constants involved, this leads us to conclude that $\dot{c} = -kH$, where k is some positive constant. Furthermore, consider the fixed point value of c. We see that it is given by $2F$. We are led to consider the linear combinations of F, G and H, such that the coefficient of F is 2, and the derivative of the combination is proportional to H. From the conservation of the energy-momentum tensor we find that Zamolodchikov's choice is the only one that has these properties.

[11] It was conjectured that this would be the case also in higher dimensions, and much work was put into the search for a more general C-function. For instance, the main goal of the article where Friedan's C-theorem proof was published was to propose such a function (Cappelli et al., 1991). However, one has now found that this assumption is not true in general. For an example of a theory with a limit cycle flow see (Glazek and Wilson, 2002) and references therein.

Ising model can never be weakly perturbed in such a way that further RG flow leads us to the 3-state Potts model, whereas the opposite process may in principle be possible.

In fact, the proof of the C-theorem has a very practical consequence. Zamolodchikov's proof hands us a recipe to find the C-function for any theory from correlators. Following Friedan, we can use spectral densities to calculate the C-function. From this we can construct RG-flow-diagrams around fixed points, and place theories into universality classes. The value of the β functions near the fixed point can be found from the C-function without much effort.

However, the concept of smearing functions used by Friedan in his proof tells us that the value of the C-function at an arbitrary point in the space of theories is scheme-dependent. In points far away from the fixed points, the only important feature of the C-function is its positivity and its negative derivative. Its actual value at any point is unimportant, only the fact that its value is smaller than c_{UV} and larger than c_{IR} is important.

The prescription for the C-theorem can be generalized to the k-theorem, as given in (Friedan, 1990/1991?). The idea of this theorem is to exchange the energy-momentum tensor for an arbitrary conserved current J_i. From the spectral representation of the autocorrelator we can construct a k-function in the same way as Friedan constructed the C-function in (Cappelli et al., 1991). This function will now not count the degrees of freedom in general, but the degrees of freedom coupled to the current J_i. The interpretation of the k-theorem is then that the degrees of freedom coupling to this current is monotonically decreasing.

5.4.1 Extensions of the C-theorem

As we have already noted, the extension of the C-theorem to more than 1 + 1 dimensions is not in general straightforward (or possible).

However, there have been attempts to construct more suitable, and possibly more physical, C-functions, for which a monotonicity theorem may exist in more than two dimensions and in non-trivial geometries.

Cappelli, Friedan and Latorre made such an effort taking advantage of the spectral decomposition, and it is in this context the Friedan proof was first published (Cappelli et al., 1991). They found a set of two candidate C-functions in the general case, but found cases of non-monotonicity of the first function, and a possible constraint to free theories on the other.

In a series of articles, (Gaite and O'Connor, 1996), (Gaite, 1998) and (Gaite, 2000), two candidate C-functions were constructed related to the entropy and relative entropy of a QFT. Both seemed promising from the point of view of their (and the C-function's) physical content of counting degrees of freedom. However, only the so-called relative entropy C-function has a good low-energy behaviour, but higher energy modifications must be made for it to be well-behaved at all energy scales.

The strenght of Gaite's approach is revealed both in its physical content and on the specific case of a cylindrical shaped geometry. Gaite (Gaite, 1998) observes that Zamolodchikov's argument does not extend to the cylinder, because the rotational symmetry upon which his proof is based, is lost in the cylindrical theory. However, observing that the 1+1 QFT with a compact time dimension of length L corresponds to a statistical mechanics model in one dimension at finite temperature β, enables us to modify the argument to that of entropy. In (Gaite, 1998) and (Gaite, 2000), this is done, and β is viewed as a new parameter. This is also the motivation for their defintions of candidate C-functions.

In (Neto and Fradkin, 1993), the authors use entropy, and a construction of entropy from the compactification of the time-dimension similar to Gaite's,[12] to say something about the restrictions of theories where a C-theorem-like monotonicity statement may be true. The conclusion is that for systems where quantum fluctuations are predominant, a C-theorem-like statement will be true. However, at the level where thermal fluctuations start taking over, such a statement will no longer be true. The reason, then, why the C-theorem is always true in the 2-dimensional case, is that the corresponding statistical mechanics model of one dimension has no thermal critical point, and hence the critical points are purely quantum driven.

In this context, it might not be suprising that a pure quantum entropy has been proposed as an interesting candidate for an extended C-theorem.

It was first shown in (Holzhey et al., 1994) that the entanglement entropy[13] S_A is proportional to the central charge at criticality.[14]

A thorough calculation of this quantity in the 2-dimensional cases, and even a proof of the C-theorem based on a combination of different entanglement entropies was given in (Calabrese and Cardy, 2004).

The article (Ryu and Takayanagi, 2006a) calculates entanglement entropies for higher dimensional geometries as well. It is interesting to note that in the 4-dimensional case, the entropy depends on two central charges c and a. Although a has the right behaviour, the central charge c does not. This splitting of a candidate C-function into two parts, only one of which possibly satisfies the C-theorem, is the same as seen in (Cappelli et al., 1991).

[12]The article (Neto and Fradkin, 1993), was one of the inspirations for the trial C-functions in (Gaite, 1998).

[13]For a nice review of entanglement entropy, see instance the second section of (Ryu and Takayanagi, 2006a).

[14]This quantity has later been used to demonstrate an intriguing relationship between (D+1)-CFT an (D+2)-AdS called holography. The papers on this are numereous, but (Ryu and Takayanagi, 2006a) gives a nice review. We will delve more deeply into this subject in chapter 10.

This again leads us to see that it is fortuitous features of 2-dimensional flat geometry that paves the way for a C-theorem. In this sense, $d = 2$ is somehow a critical dimension for the RG-flow.

Chapter 6

Ising on the Plane

> We wish to reassert the picture plane.
> We are for flat forms because they destroy illusion and reveal truth.

From point 4 of Mark Rothko, Adolph Gottlieb and Barnett Newman's *brief manifesto* printed in the June 13, 1943 edition of the New York Times

In this part we will look at a free Majorana field in two dimensions on $\mathbb{R}^2$, that is on the flat 2-dimensional plane. The reason for looking at this particular model is its behaviour as we approach criticality, there this model is the CFT of the Ising model.

We will calculate the C-function on the plane and see that it has the properties described in the C-theorem. In the following chapters we will then move on to more complicated 2-dimensional geometries where one or more dimension is compactified. Here we will calculate the contributions to the vacuum energy from the compactification.

The free Majorana field has the action

$$S = \int dr^2 \left(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi} + im\overline{\psi}\psi \right) . \tag{6.1}$$

6.1 The Energy-Momentum Tensor

To find the C-function we need the energy-momentum tensor. We have:[1]

$$dS = -\frac{1}{2\pi} \int T_{\mu\nu}(r)\partial^{\mu}\alpha^{\nu}(r)dr^2 \tag{6.2}$$

Thinking of the $\mathbb{R}^2$ plane in terms of a 1-dimensional complex plane rather than a 2-dimensional real one, we now use z and $\overline{z}$ as independent

[1] See for instance (Cardy, 1990).

variables:

$$r^z = z, \qquad r^{\overline{z}} = \overline{z}, \qquad \psi_z = \psi, \qquad \psi_{\overline{z}} = \overline{\psi}.$$

To find dS we perform the transformation $r^\mu \to r^\mu + \alpha^\mu(r)$ and we get:

$$dr^2 \to dr^2 \left(1 + 2\partial^\mu \alpha^\nu + O(\partial^2)\right) \tag{6.3}$$

Subtracting the original S and including only first order in derivatives, we arrive at the following expression for dS:

$$dS = \frac{1}{2\pi} \sum_{\mu,\nu} \int dr^2 \left(\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi} + im\overline{\psi}\psi\right) \partial^\mu \alpha^\nu(r) \tag{6.4}$$

Observing that the $\partial^\mu \alpha^\nu$ terms only give rise to non-zero terms when combined with non-zero $\psi_\mu \psi_\nu$ terms, we find that the stress-energy tensor has the following components:

$$\begin{aligned} T = T_{zz} &= -\pi \psi \overline{\partial} \psi && (6.5) \\ \overline{T} = T_{\overline{z}\overline{z}} &= -\pi \overline{\psi} \partial \overline{\psi} && (6.6) \\ \Theta = 4T_{z\overline{z}} = 4T_{\overline{z}z} &= 2\pi i m \overline{\psi}\psi. && (6.7) \end{aligned}$$

We can now use equations (6.5), (6.6) and (6.7) to find the C-function by virtue of the following relations between correlations of the stress-energy tensor components and the functions F, G, H and C, as defined in chapter 5:

$$\langle T(z,\overline{z})T(0,0)\rangle = (\pi)^2 \langle \psi(z,\overline{z})\overline{\partial}\psi(z,\overline{z})\psi(0,0)\overline{\partial}\psi(0,0)\rangle = \frac{F(z,\overline{z})}{z^4} \tag{6.8}$$

$$\begin{aligned} \langle \Theta(z,\overline{z})T(0,0)\rangle &= \langle T(z,\overline{z})\Theta(0,0)\rangle \\ &= -2i\pi^2 m \langle \psi(z,\overline{z})\overline{\partial}\psi(z,\overline{z})\overline{\psi}(0,0)\psi(0,0) = \frac{G(z,\overline{z})}{z^3\overline{z}}\rangle \end{aligned} \tag{6.9}$$

$$\langle \Theta(z,\overline{z})\Theta(0,0)\rangle = -4\pi^2 m^2 \langle \overline{\psi}(z,\overline{z})\psi(z,\overline{z})\overline{\psi}(0,0)\psi(0,0)\rangle = \frac{H(z,\overline{z})}{z^2\overline{z}^2} \tag{6.10}$$

$$C \equiv 2F - G - \frac{3}{8}H \tag{6.11}$$

In order to calculate the C-function from these relations, all we need is a few correlation functions, a few Bessel functions and Wick's theorem. A description of the Bessel functions, how they arise in this problem, and a short list of their properties can be found in Appendix IV.

6.2 Some Preliminary Correlation Functions

We use knowledge of Bessel functions to calculate the correlation functions. Together with Wick's theorem (and Poincaré invariance), this will give us F, G and H.

Define $R \equiv (z\overline{z})^{1/2} = |z|$. We also employ the abbreviation $\psi(z, \overline{z}) \equiv \psi(z)$.

We find the correlations by taking derivatives of the partition function with respect to the (primary) operators we would like to find correlation functions of, e.g.:

$$\langle\overline{\psi}(z)\psi(0)\rangle = -im\int \frac{d^2p}{(2\pi)^2}\frac{e^{\frac{i}{2}(p\overline{z}+\overline{p}z)}}{p^2+m^2} = -\frac{im}{2\pi}K_0(mR) = -\langle\psi(z)\overline{\psi}(0)\rangle \tag{6.12}$$

$$\langle\psi(z)\psi(0)\rangle = -i\int \overline{p}\frac{d^2p}{(2\pi)^2}\frac{e^{\frac{i}{2}(p\overline{z}+\overline{p}z)}}{p^2+m^2} = 2\partial_{\overline{z}}\frac{1}{2\pi}K_0(mR) = -m\frac{\sqrt{\overline{z}/z}}{2\pi}K_1(mR) \tag{6.13}$$

$$\langle\overline{\psi}(z)\overline{\psi}(0)\rangle = 2\partial_z\frac{1}{2\pi}K_0(mR) = -m\frac{\sqrt{z/\overline{z}}}{2\pi}K_1(mR) \tag{6.14}$$

From now on $K_0(mR) \equiv mR$ and $K_1(mR) \equiv K_1$ to avoid confusion with factors of mR:

$$\begin{aligned}
\langle\overline{\partial}\psi(z)\psi(0)\rangle &= \overline{\partial}\langle\psi(z)\psi(0)\rangle \\
&= \partial_{\overline{z}}\left(-m\frac{\sqrt{\overline{z}/z}}{2\pi}K_1\right) \\
&= -\frac{m}{2\pi}\left(-\frac{1}{2}\sqrt{\frac{\overline{z}}{z^3}}K_1 + \frac{m}{2}\frac{\overline{z}}{z}\left(-K_0 - \frac{K_1}{mR}\right)\right) \\
&= \frac{m^2}{4\pi}\frac{\overline{z}}{z}\left(K_0 + \frac{2K_1}{mR}\right)
\end{aligned} \tag{6.15}$$

By Poincaré invariance the above is also equal to $-\langle\psi(z)\overline{\partial}\psi(0)\rangle$. Similarly,

$$\begin{aligned}
\langle\overline{\partial}\psi(z)\overline{\psi}(0)\rangle &= \overline{\partial}\langle\psi(z)\overline{\psi}(0)\rangle \\
&= \overline{\partial}\left(\frac{im}{2\pi}K_0\right) \\
&= -\frac{im}{2\pi}K_1\frac{m}{2}\sqrt{\frac{\overline{z}}{z}} \\
&= -\frac{im^2}{4\pi}K_1\sqrt{\frac{\overline{z}}{z}}
\end{aligned} \tag{6.16}$$

and finally:

$$\begin{aligned}
\langle\overline{\partial}^2\psi(z)\psi(0)\rangle &= -\langle\overline{\partial}\psi(z)\overline{\partial}\psi(0)\rangle \\
&= \overline{\partial}\left(\frac{m^2}{4\pi}\frac{\overline{z}}{z}\left(K_0+\frac{2K_1}{mR}\right)\right) \\
&= \frac{m^2}{4\pi}\overline{\partial}\left(2\sqrt{\frac{\overline{z}}{z^3}}\frac{K_1}{m}+\frac{\overline{z}}{z}K_0\right) \\
&= \frac{m^2}{4\pi}\Bigg(-3\sqrt{\frac{\overline{z}}{z^5}}\frac{K_1}{m}+\frac{2}{m}\frac{\overline{z}}{z^2}\left(-K_0-\frac{K_1}{mR}\right)\frac{m}{2} \\
&\quad -\frac{\overline{z}}{z^2}K_0-\frac{\overline{z}}{z}K_1\frac{m}{2}\sqrt{\frac{\overline{z}}{z}}\Bigg) \\
&= -\frac{m^2}{4\pi}\frac{\overline{z}}{z^2}\left(2K_0+\frac{4K_1}{mR}+\frac{K_1 mR}{2}\right) \qquad (6.17)
\end{aligned}$$

6.3 F, G, H ... and C

Now we are ready to calculate F, G and H.

6.3.1 $H(z,\overline{z})$

We start by finding H. Applying Wick's theorem to (6.10) we get:

$$\langle\Theta(z,\overline{z})\Theta(0,0)\rangle = -4\pi^2m^2\Big(|\langle\psi(z)\overline{\psi}(0)\rangle|^2-\langle\overline{\psi}(z)\overline{\psi}(0)\rangle\langle\psi(z)\psi(0)\rangle\Big) \qquad (6.18)$$

Applying the results from equations (6.12), (6.14) and (6.13) yields:

$$\begin{aligned}
\langle\Theta(z,\overline{z})\Theta(0,0)\rangle &= -4\pi^2m^2\Bigg(\frac{-im}{2\pi}K_0\frac{im}{2\pi}K_0- \\
&\quad (-m)\frac{\sqrt{\overline{z}/z}}{2\pi}K_1(-m)\frac{\sqrt{z/\overline{z}}}{2\pi}K_1\Bigg) \qquad (6.19)
\end{aligned}$$

After multiplying and summing up, the result is:

$$\langle\Theta(z,\overline{z})\Theta(0,0)\rangle = m^4\left(K_1^2-K_0^2\right) \qquad (6.20)$$

Multiplying equation (6.10) by $z^2\overline{z}^2$ gives us the value of H:

$$H(z,\overline{z}) = (mR)^4\left(K_1^2-K_0^2\right) \qquad (6.21)$$

As expected, $H\to 0$ at criticality ($m\to 0$).

6.3.2 $G(z,\overline{z})$

Equation (6.9) gives us that:

$$\begin{aligned}\langle T(z,\overline{z})\Theta(0,0)\rangle &= -2i\pi^2 m\Big(\langle\psi(z)\psi(0)\rangle\langle\overline{\partial}\psi(z)\overline{\psi}(0)\rangle - \\ &\qquad \langle\psi(z)\overline{\psi}(0)\rangle\langle\overline{\partial}\psi(z)\psi(0)\rangle\Big) \qquad (6.22)\end{aligned}$$

Substituting the correlation functions for the values we found in equations (6.13), (6.16), (6.12) and (6.15) we get:

$$\begin{aligned}\langle T(z,\overline{z})\Theta(0,0)\rangle &= -2i\pi^2 m\left((-m)\frac{\sqrt{\overline{z}/z}}{2\pi}K_1\frac{-im^2}{4\pi}K_1\sqrt{\frac{\overline{z}}{z}} - \right. \\ &\qquad \left. \frac{im}{2\pi}K_0\frac{m^2}{4\pi}\frac{\overline{z}}{z}\left(K_0+\frac{2K_1}{mR}\right)\right) \qquad (6.23)\end{aligned}$$

and after a few lines of arithmetic:

$$\langle T(z,\overline{z})\Theta(0,0)\rangle = \frac{m^4}{4}\frac{\overline{z}}{z}\left(K_1^2-K_0^2-\frac{2K_1K_0}{mR}\right) \qquad (6.24)$$

Then multiplying equation (6.9) by $z^3\overline{z}$, we obtain:

$$G(z,\overline{z}) = \frac{(mR)^4}{4}\left(K_1^2-K_0^2-\frac{2K_1K_0}{mR}\right) \qquad (6.25)$$

Observe that $G(z,\overline{z})\to 0$ at criticality, as is expected since in a trivial geometry $\Theta\to 0$ at criticality.

6.3.3 $F(z,\overline{z})$

We again make use of Wick's theorem, this time applying it to equation(6.8).

$$\langle T(z,\overline{z})T(0,0)\rangle = \pi^2\left(-\left(\overline{\partial}\langle\psi(z)\psi(0)\rangle\right)^2+\left(\langle\psi(z)\psi(0)\rangle\overline{\partial}^2\langle\psi(z)\psi(0)\rangle\right)\right) \qquad (6.26)$$

Applying the results from (6.15), (6.13) and (6.17) yields:

$$\begin{aligned}\langle T(z,\overline{z})T(0,0)\rangle &= \pi^2\left[-\left(\frac{m^2}{4\pi}\frac{\overline{z}}{z}\left(K_0+\frac{2K_1}{mR}\right)\right)^2+\right. \\ &\qquad (-m)\frac{\sqrt{\overline{z}/z}}{2\pi}K_1(mR)\left(\frac{-m^2}{4\pi}\frac{\overline{z}}{z^2}\right. \\ &\qquad \left.\left.\times\left(2K_0+\frac{4K_1}{mR}+\frac{K_1 mR}{2}\right)\right)\right] \qquad (6.27)\end{aligned}$$

A few lines of manipulations then lead to

$$\langle T(z,\overline{z})T(0,0)\rangle = \frac{m^4}{16}\left(\frac{\overline{z}}{z}\right)^2 \left(K_1^2 - K_0^2 + 4\frac{K_1^2}{(mR)^2}\right) \tag{6.28}$$

Multiplying (6.8) by z^4 then gives F:

$$F(z,\overline{z}) = \frac{(mR)^4}{16}\left(K_1^2 - K_0^2 + 4\frac{K_1^2}{(mR)^2}\right) \tag{6.29}$$

Using the critical behaviour of K_0 and K_1 as $m \to 0$ given in equations (IV.17) and (IV.18), we see that

$$\lim_{m\to 0}\left(\frac{F}{z^4}\right) = \frac{1}{4}\frac{1}{z^4}, \tag{6.30}$$

which is as expected.

6.3.4 The C-function

With the results from the three previous sections, we are now ready to calculate the C-function. We apply (6.30), (6.25) and (6.21) to (6.11):

$$\begin{aligned} C \;=\; & \frac{(mR)^4}{16}\Big[2\left(K_1^2 - K_0^2 + 4\frac{K_1^2}{(mR)^2}\right) \\ & -4\left(K_1^2 - K_0^2 - \frac{2K_1K_0}{mR}\right) - 6\left(K_1^2 - K_0^2\right)\Big], \end{aligned} \tag{6.31}$$

which eventually leads to

$$C = \frac{(mR)^4}{2}\left(K_0^2 - K_1^2 + \frac{K_1^2}{(mR)^2} + \frac{K_1K_0}{mR}\right). \tag{6.32}$$

We observe that the behaviour at criticality is

$$\lim_{m\to 0} C = \frac{1}{2}, \tag{6.33}$$

which is the central charge of the Ising model.

In addition to the critical behaviour, we may also point out that the functions F, G, H and C only depend on z and $\overline{z}$ through R. This is a general result which applies to all correlation functions in geometries with rotational and translational symmetry.

A plot of the C-function as a function of the absolute value of mR is shown in figure 6.1.

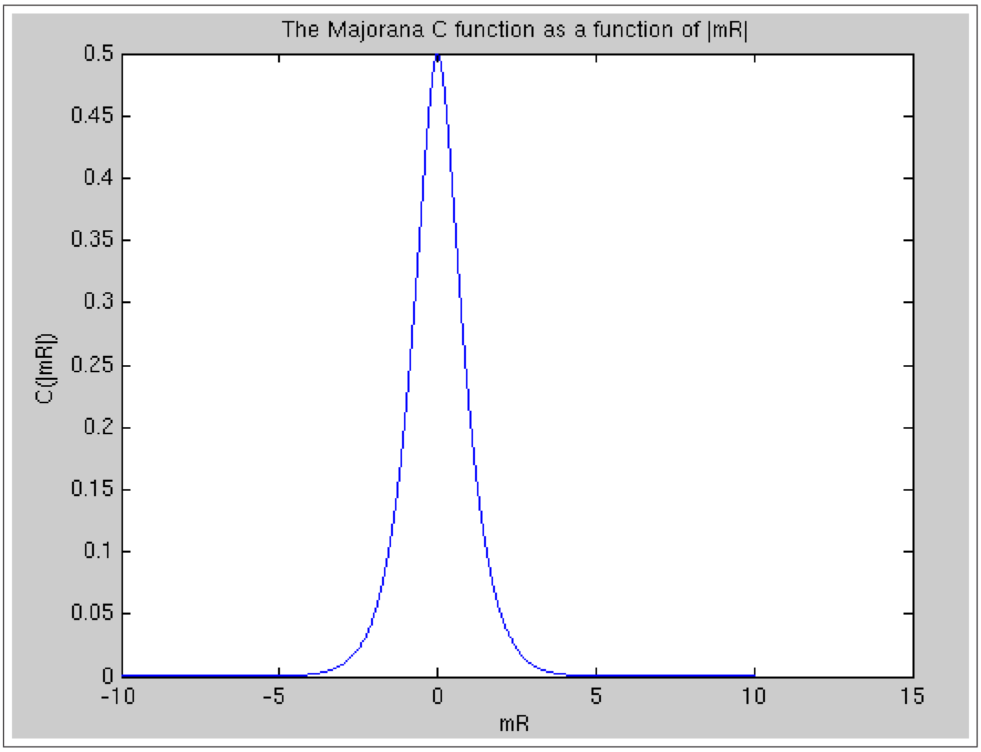

Figure 6.1: A plot of the Majorana C-function as a function of $|mR|$ as given by equation (6.32). This is a kind of RG-flow in the 1-dimensional parameter space (m) where the flow is directed in the direction of decreasing C-values. As expected, the functioin is flat (i.e. the β-function vanishes) at the critical/massless point.

Chapter 7

Ising on the Cylinder: The Casimir Effect in Action

Treat nature in terms of the cylinder, the sphere, and the cone,...

Paul Cézanne as quoted in *What he told me –The motif*, from (Gasquet, 1991).

The correspondence between QFT and quantum statistical mechanics is very transparent in the path integral formalism of QFT with the correspondance $it \to \beta$ were β is the inverse temperature.[1] This yields a mapping between a d-dimensional statistical mechanics theory, and a $d+1$-dimensional QFT. Finite temperature corresponds to a compactification of the time direction, hence a cylinder. A cylinder of width L in the time direction then corresponds to a temperature of $T = L^{-1}$. This is one of the reasons why the cylinder geometry is interesting (Gaite, 2000).

7.1 Transformation from the Plane to the Cylinder

Before we start calculating the vacuum energy of the 2-dimensional cylinder for the Majorana fermions, we would like to see what the transformation from a plane to a cylinder does to a general CFT. We do this by transforming onto the cylinder by a conformal transformation.

Mapping a theory on a plane to a theory on a cylinder ($-\infty < u < \infty, 0 \leq v \leq l$) can be achieved by a conformal transformation[2]

$$w \equiv u + iv = \frac{l}{2\pi} \ln z. \tag{7.1}$$

This means that the conformal field theory stays conformal when it is moved from the plane to the cylinder.

[1] See for instance (Di Francesco et al., 1997) or (Peskin and Schroeder, 1995).

[2] See for instance (Cardy, 1990).

We can use this to find the vacuum energy as done in (Blöte et al., 1986). The same result was also derived in (Affleck, 1986), but we will follow the line of argument in (Cardy, 1990), which is close to the derivation of (Blöte et al., 1986).

Recall from chapter 4 that the transformation of a scaling or primary operator under a finite conformal transformation $z \to z' = f(z)$ is given by

$$\phi(z, \overline{z}) \to \left(f'(z)\right)^{-h} \left(\overline{f'(z)}\right)^{-\overline{h}} \phi(z', \overline{z}'), \tag{7.2}$$

where h and $\overline{h}$ are the conformal dimensions of the scaling operator.

The diagonal components of the energy-momentum tensor T and $\overline{T}$ have the scaling dimensions $(2, 0)$ and $(0, 2)$ respectively. However, the components of the energy-momentum tensor are secondary operators and their definition prohibits an arbitrary normalisation. This yields an anomalous term in the transformation of T and $\overline{T}$:

$$T(z) \to f'(z)^2 T(z') + \frac{c}{12}\{z', z\} \tag{7.3}$$

$$\overline{T}(z) \to \overline{f'(z)}^2 \overline{T}(\overline{z}') + \frac{c}{12}\{\overline{z}', \overline{z}\} \tag{7.4}$$

where $\{z', z\}$ denotes the Schwartzian derivative:

$$\{z', z\} = \frac{f''' f' - 3f''^2/2}{f'^2}. \tag{7.5}$$

Entering our specific conformal transformation into these transformation properties yields:

$$T(w)_{\text{cyl}} = \left(\frac{2\pi}{l}\right)^2 \left(z^2 T(z)_{\text{plane}} - \frac{c}{24}\right).$$

We now find the Hamiltonian and momentum operators from

$$\widehat{H} = \frac{1}{2\pi} \int_0^l \widehat{T}_{uu}(v) dv \qquad \text{and} \qquad \widehat{P} = \frac{1}{2\pi} \int_0^l \widehat{T}_{uv}(v) dv.$$

Changing integration variables to z and $\overline{z}$ and making use of the Virasoro generators

$$L_0 = \frac{1}{2\pi i} \oint z T(z) dz, \qquad \overline{L}_0 = -\frac{1}{2\pi i} \oint \overline{z}\,\overline{T}(\overline{z}) d\overline{z}, \tag{7.6}$$

we find:

$$\widehat{H} = \frac{2\pi}{l}(L_0 + \overline{L}_0) - \frac{\pi c}{6l} \qquad \text{and} \qquad \widehat{P} = \frac{2\pi}{l}(L_0 - \overline{L}_0)$$

We see that the eigenstates of the Hamiltonian and momentum operators are identical to the eigenstates of $(L_0, \overline{L}_0)$. Their eigenvalues are

$$E = E_0 + \frac{2\pi x}{l}, \qquad k = \frac{2\pi s}{l}, \tag{7.7}$$

where $x = h + \overline{h}$ and $s = h - \overline{h}$, and $E_0 \equiv -\pi c/6l$.

7.2 Vacuum Energy

We read off the vacuum energy of a conformal field theory on the cylinder directly from equation (7.7).

$$E_0^c = -\frac{\pi c}{6l}. \tag{7.8}$$

This means that for a conformal field theory of central charge $c = 0$, there is no vacuum energy or Casimir effect due to the compactification of one dimension. This should come as no suprise, since the C-function on the plane, and hence the value of the conformal charge at the fixed points on the cylinder, counts the degrees of freedom of the theory. A theory of no degrees of freedom should not have vacuum energy, whatever the length of the cylinder upon which it resides.

So far the general conformal case. Specializing to Majorana fermions the conformal situation is then the Ising model with $c = 1/2$. We see that the vacuum energy for massless Majorana fermions on a cylinder of width l is

$$E_0^{c=\frac{1}{2}} = -\frac{\pi}{12l}. \tag{7.9}$$

We will now try to extend the notion of this vacuum energy to the non-conformal case. We will follow the arguments found in (Gaite, 2000). In his article, he defines thermodynamic quantities like entropy and free energy in a very general way. He finds that the free energy takes the form of a bulk term and a finite-size correction (due to the compactification of one dimension):

$$\frac{-\ln Z}{L} = \beta\frac{F}{L} = e_0(\Lambda, \lambda)\beta + \frac{C(\beta, \lambda)}{\beta}, \tag{7.10}$$

where L is the length of the cylinder,[3] β is the circumference of the cylinder formerly known as l,[4] Λ is a renormalisation scale, and λ is a collection of coupling constants.

[3] L is formally to be taken to infinity in our case. However, the vacuum energy should be measured per length of the cylinder.

[4] The reason for the change of name is the identification of β as equivalent to an inverse of a finite temperature in the case of a compactification of the time direction in a correspondence between a $(1 + 1)$-dimensional quantum field theory, and a 1-dimensional statistical mechanical theory.

The e_0 part is the bulk free energy, and essentially not different from the plane value in a sufficiently large cylinder. The other part C/β is the finite-size correction, and hence the vacuum energy, due to the compactification.

Gaite calculates the value of the free energy explicitly for the Gaussian and Ising models. Our case of the Majorana fermions is easily identified with the Ising case, which gives:[5]

$$\beta\frac{F}{L} = e_0\beta \mp \frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}(\pm)^n\left[K_2(nm\beta) - K_0(nm\beta)\right], \tag{7.11}$$

where the upper signs correspond to the Gaussian case whereas the lower signs belong to the Ising/Majorana case.

Using our knowledge of the Bessel functions:[6]

$$E_0 = \mp\frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}(\pm)^n\frac{2}{m\beta n}K_1(m\beta n) \tag{7.12}$$

We can check that these results give the correct values in the conformal case ($m \to 0$) by inserting the limit value of $K_1(x)$ when $x \to 0$.

For comparison we do the insertion both for the Gaussian and Ising cases:

$$\begin{aligned} E_0^{c=1} &= -\lim_{m\to 0^+}\frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}\frac{2}{m\beta n}K_1(m\beta n) \\ &= -\lim_{m\to 0^+}\frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}\frac{2}{(m\beta n)^2} \\ &= -\frac{1}{\pi\beta}\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &= -\frac{1}{\pi\beta}\left(\frac{\pi^2}{6}\right) \\ &= -\frac{\pi\cdot 1}{6\beta} \end{aligned} \tag{7.13}$$

which is just what we expected since $c = 1$ in the Gaussian (bosonic) case.

[5] A detailed derivation of the result below can of course be found in (Gaite, 2000).

[6] See Appendix IV, specifically equation (IV.13).

For the Ising model:

$$
\begin{aligned}
E_0^{c=\frac{1}{2}} &= \lim_{m\to 0^+} \frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2}{m\beta n}K_1(m\beta n) \\
&= \lim_{m\to 0^+} \frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2}{(m\beta n)^2} \\
&= \frac{1}{\beta\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2} \\
&= \frac{1}{\beta\pi}\left(\frac{\pi^2}{6}-\frac{\pi^2}{2}+\frac{\pi^2}{4}\right) \\
&= -\frac{\pi\cdot\frac{1}{2}}{6\beta} = -\frac{\pi}{12\beta} \qquad (7.14)
\end{aligned}
$$

which again confirms our value of $c = 1/2$.[7]

For later convinience we would like to find an expansion in m of $E_0^{c=\frac{1}{2}}$. In (Gaite, 2000) such an expansion is given up to m^2. Subtracting a cut-off dependent logarithmic divergence which may be absorbed by the non-universial bulk term e_0, we find:

$$
E_0^{c=\frac{1}{2}} = -\pi\left[\frac{1}{12\beta} - \beta\left(\frac{m}{2\pi}\right)^2\left(-\gamma+\frac{1}{2}\right)\right], \qquad (7.15)
$$

where γ is Euler's constant as given in equation (III.6) of Appendix III.1.

[7]The second and third steps can be found in for instance (Rottmann, 1995).

Chapter 8

Ising on the Torus: Modular Invariance - a Useful Straight Jacket

> Whether you take the doughnut hole as a blank space or as an entity unto itself is a purely metaphysical question and does not affect the taste of the doughnut one bit.
>
> Haruki Murakami in *A Wild Sheep Chase* (Murakami, 1989)

In this section we will study our Majorana theory on the torus. Our goal is to find the vacuum energy in this geometry. This is, however, far from straightforward.

We will therefore begin with the calculation at the critical point and deduce the partition function from a set of boundary conditions (called sectors) and modular invariance. A more or less general discussion on these constraints can be found in most texts on conformal field theory. My primary sources on this subject have been (Di Francesco et al., 1997), (Cardy, 1990) and (Itzykson and Drouffe, 1989).

Thereafter, we will follow (Saluer and Itzykson, 1987), using our knowledge of the combination of sectorwise contributions to find the partition function in the massive case.

8.1 Torus Boundary Conditions - Sectors and Modular Invariance

Recal+l that in (Gaite, 1998, 2000) the propagator was obtained, and from it all correlation functions, for the Ising model/Majorana fermions on the cylinder, by compactifying one dimension and imposing anti-periodic boundary conditions on a Gaussian model.

We will view the theory on the torus in much the same way. Starting from the same $\mathbb{R}^2$ propagator, we will compactify both dimensions. However, since two directions are now compact, we must choose between four different sets of boundary conditions. We call these sets sectors.[1]

The sectors are given by the transformation properties of the fields:

$$\psi(z+\omega_1) = e^{2i\pi v}\psi(z), \qquad \psi(z+\omega_2) = e^{2i\pi u}\psi(z),$$

where $\omega_{1,2}$ are the sides of the parallelogram the torus is built from, so that the parameter $\tau = \omega_2/\omega_1$.[2]

u and v can take the values of 0 or $1/2$, hereby defining the sectors. We can denote each sector by its values of u or v, although it is also customary to call a periodic boundary condition (0) a Ramond condition (R for short) and an anti-periodic (1/2) Neveu-Schwartz (NS for short). The different sectors are then:

$$(v,u) = (0,0) \quad (R,R) \qquad (v,u) = \left(\frac{1}{2}, 0\right) \quad (NS,R)$$

$$(v,u) = \left(0, \frac{1}{2}\right) \quad (R,NS) \qquad (v,u) = \left(\frac{1}{2}, \frac{1}{2}\right) \quad (NS,NS).$$

In section 3.2.3 the conditions of modular invariance were introduced, and a justification for them given. We will now use these conditions to impose constraints on our theory.

We see that the four sectors do not mix, so they all give seperate propagators and partition functions. In general, however, we expect the full partition function to be a linear combination of the partition functions of the different sectors. The conditions of modular invariance will then hopefully limit the possible values of such linear combinations.

We assume then that the form of the partition function will be

$$Z_{\text{Ising}} = AZ_{\frac{1}{2},\frac{1}{2}} + BZ_{0,\frac{1}{2}} + CZ_{\frac{1}{2},0} + DZ_{0,0} \tag{8.1}$$

where $Z_{v,u}$ is the partition function for the sector (v,u) and A, B, C and D are constants to be fixed by modular invariance.[3]

[1] The different sectors are the different spin-structures on the torus, and it would be more correct to call them spin-structures. However, in accordance with the notation in (Di Francesco et al., 1997) we use the word sector.

[2] See section 3.2.3 for a description of the torus.

[3] Possibly supplemented by some normalization.

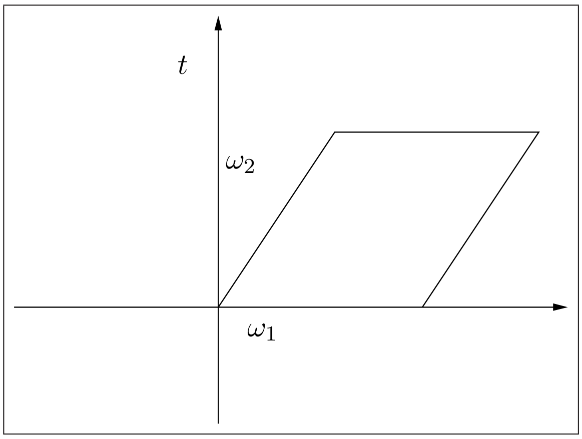

Figure 8.1: The torus with periods ω_1 and ω_2, $\tau = \omega_2/\omega_1$. In the normalized case $\omega_1 = 1$ and $\omega_2 = \tau$.

8.2 The Conformal Theory on the Torus

In the conformal case, we can derive the partition functions of the different sectors in a purely algebraic fashion. Imposing the constraint of modular invariance on equation (8.1) on the algebraically derived partition functions will then give us the conformal partition function, which is a good starting point for the massive case calculation.

We can find the partition function in many different ways, including the functional integral way. However, in the conformal case an algebraic method taking advantage of our knowledge of the Virasoro generators will be easier, and therefore our chosen path. This derivation can also be found in (Di Francesco et al., 1997). We reproduce it here for completeness.

We start by introducing a transfer matrix. The transfer matrix translates the system parallell to one of the torus periods, and we can therefore automatically impose boundary conditions by the use of it. We choose to make our translations parallell to the period ω_2, because this is the period that has a component in the time-direction.[4]

In the physical world the time direction is different from the space direction. We identify the time direction with the imaginary axis and the space direction with the real one. Moving to a Euclidian picture, we let $it \to s$.

Now we need only identify the generators of translation along time and space directions. These are the Hamiltonian H and the momentum operator P, respectively.

If the transfer matrix T_a translates the system a distance a along the ω_2

[4]See figure 8.1.

direction, we can easily see that it should be given as:

$$T_a = e^{-a(H\Im\omega_2 - iP\Re\omega_2)/|\omega_2|} \tag{8.2}$$

In a statistical mechanical system, a can be a lattice spacing. In a field theoretical distcription a may be a UV regulator.

The partition function $Z(\omega_1, \omega_2)$ is the trace of the full circle translation. That is, if $|\omega_2| = ma$ where $m \in \mathbb{N}$, then Z is the trace of T_a^m. Applying this to equation (8.2):

$$Z(\omega_1, \omega_2) = \text{Tr}e^{-(H\Im\omega_2 - iP\Re\omega_2)}. \tag{8.3}$$

From conformal field theory, we know that H and P can be given in terms of the Virasoro generators on a cylinder of circumference L:

$$\begin{aligned} H &= \frac{2\pi}{L}\left(L_0 + \overline{L_0} - \frac{c}{12}\right) \\ P &= \frac{2\pi}{L}\left(L_0 - \overline{L_0}\right) \end{aligned} \tag{8.4}$$

Our torus was made from a cylinder of circumference $L = \omega_1$. Anticipating modular invariance and moving to our main representative of the equivalence class of tori of parameter τ, we insert that $\omega_1 = 1$ and $\omega_2 = \tau$, so that $\Im\omega_2 = (\tau - \overline{\tau})/(2i)$ and $\Re\omega_2 = (\tau + \overline{\tau})/2$. Inserting this into (8.3) yields:

$$Z(\tau) = \text{Tr}e^{2\pi i\left(\tau(L_0 - c/24) - \overline{\tau}(\overline{L_0} - c/24)\right)}$$

$$Z(\tau) = \text{Tr}q^{L_0 - c/24}\overline{q}^{\overline{L_0} - c/24} \tag{8.5}$$

where $q = e^{2\pi i\tau}$, $\overline{q} = e^{-2\pi i\overline{\tau}}$.

Equipped with the rather general expression in equation (8.5) for the partition function, we impose the boundary conditions for each sector. Since we are dealing with fermions, taking a full translation of one fermion ψ in a correlator $< \psi X >$, with X a collection of an odd number of fermions, should give a minus sign since the fermion is then moved past all the other fermions by time ordering.

This minus sign enters naturally in the sectors where the boundary conditions are of the type $(v, 1/2)$, since the boundary condition on ψ here does the job for us.

However, in the $(v, 0)$ sectors we need to add an operator that does this manually. Such an operator has to anti-commute with all fermions. One operator of this kind is $(-1)^F$, where

$$F = \sum_{k \geq 0} F_k \qquad F_k = b_{-k}b_k(k > 0),$$

and b_k are the usual fermion creation and annihilation operators. F_0 annihilates the vacuum and yields 1 when acting on $b_0|0>$.

Furthermore, the operator L_0 should be shifted by $+c/8 = 1/16$ for the sectors $(0,u)$ which are periodic in the space direction. The expressions for L_0 then become:

$$L_0 = \sum_k kb_{-k}b_k, \qquad (\frac{1}{2}, u) \tag{8.6}$$

$$L_0 = \sum_k kb_{-k}b_k + \frac{1}{16}, \qquad (0, u) \tag{8.7}$$

We are almost ready to calculate the partition function for each sector.[5]

$$Z_{u,v} = d_{u,v}\overline{d}_{u,v}$$

Observe that the holomorphic and anti-holomorphic parts do not mix, and that they are simply the complex conjugates of each other, so it is sufficient to calculate the holomorphic part.

$$d_{0,0} = \frac{1}{\sqrt{2}}\mathrm{Tr}(-1)^F q^{\sum_k kb_{-k}b_k+\frac{1}{24}}$$

$$d_{0,\frac{1}{2}} = \frac{1}{\sqrt{2}}\mathrm{Tr}q^{\sum_k kb_{-k}b_k+\frac{1}{24}}$$

$$d_{\frac{1}{2},0} = \mathrm{Tr}(-1)^F q^{\sum_k kb_{-k}b_k-\frac{1}{48}}$$

$$d_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}q^{\sum_k kb_{-k}b_k-\frac{1}{48}}$$

The normalisation factors are conventional (Di Francesco et al., 1997). Observe that $\mathrm{Tr}q^{kb_{-k}b_k} = 1 + q^k$, $\mathrm{Tr}(-1)^F q^{kb_{-k}b_k} = 1 - q^k$.

Using this and the definitions of the Jacobi θ functions and the Dedekind η function given in equations (V.7), (V.8), (V.9) and (V.10) of Appendix (V.1), we see that:

$$d_{0,0} = \frac{1}{\sqrt{2}}q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^n) = 0 \tag{8.8}$$

$$d_{0,\frac{1}{2}} = \frac{1}{\sqrt{2}}q^{\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^n) = \sqrt{\frac{\theta_2(0|\tau)}{\eta(\tau)}} \tag{8.9}$$

[5] We could possibly imagine a partition function where the anti-holomorphic parts were completely erased or entered in some non-trivial way. However, the phase factors in the modular transformations of the θ functions demand that they enter as modulus squared. Another reason for their simple modulus squared form can be found in the anticipation of a massive theory outside the critical point. We will then have a coupling of the holomorphic and anti-holomorphic parts in the mass term, hence we expect their contributions to the partition functions to be multiplied by each other.

$$d_{\frac{1}{2},0} = q^{-\frac{1}{48}} \prod_{n=\frac{1}{2}}^{\infty} (1 - q^n) = \sqrt{\frac{\theta_4(0|\tau)}{\eta(\tau)}} \tag{8.10}$$

$$d_{\frac{1}{2},\frac{1}{2}} = q^{-\frac{1}{48}} \prod_{n=\frac{1}{2}}^{\infty} (1 + q^n) = \sqrt{\frac{\theta_3(0|\tau)}{\eta(\tau)}} \tag{8.11}$$

where $\theta_\nu(z|\tau)$ are the Jacobi θ functions, and $\eta(\tau)$ is Dedekind's η function.

It remains to impose the modular invariance constraint on :

$$Z = A|d_{0,0}|^2 + B|d_{0,\frac{1}{2}}|^2 + C|d_{\frac{1}{2},0}|^2 + D|d_{\frac{1}{2},\frac{1}{2}}|^2 \tag{8.12}$$

In Appendix V.1 the required properties are given, and from equations (V.11) through (V.18) we must have $B = C = D$.[6] This leaves only a normalization, which is $1/2$ in the Ising case.[7] In summary:

$$Z_{\text{Ising}} = \frac{|\theta_2(0|\tau)| + |\theta_3(0|\tau)| + |\theta_4(0|\tau)|}{2|\eta(\tau)|}. \tag{8.13}$$

8.2.1 The vacuum energy

From the partition function calculated above, we can derive all physical quantities of the theory. In particular, we wish to find the vacuum energy as a function of τ (q).

To derive an expression for the vacuum energy on the torus, we recall how we started our derivation of the partition function. Since the vacuum energy is $\langle H \rangle$, we see from (8.3) that

$$E_0 = -\frac{\partial \ln Z}{\partial \Im \omega_2}. \tag{8.14}$$

Now we need to find the expression for this in terms of τ and $\overline{\tau}$. First we observe that due to the modular invariance of the theory, we may use

[6]The value of A may seem irrelevant since this sector does not contribute at all. However, in the massive case this part is not zero. Anticipating this, we keep the A.

[7]We will not show this, see (Di Francesco et al., 1997).

our usual representative torus, e.g. $\Im\omega_2 = \Im\tau$. Equation (II.6) gives:[8]

$$\begin{aligned} E_0 &= -\frac{i}{Z}(\partial_\tau - \partial_{\overline{\tau}})Z \qquad (8.15)\\ &= \frac{-i}{Z}\Bigg((\partial_\tau - \partial_{\overline{\tau}})|d_{0,0}|^2 + (\partial_\tau - \partial_{\overline{\tau}})|d_{0,\frac{1}{2}}|^2 \\ &\quad +(\partial_\tau - \partial_{\overline{\tau}})|d_{\frac{1}{2},0}|^2 + (\partial_\tau - \partial_{\overline{\tau}})|d_{\frac{1}{2},\frac{1}{2}}|^2\Bigg) \qquad (8.16)\end{aligned}$$

We now find the holomorphic derivatives sector by sector. The anti-holomorphic counterpart is totally analogous. We will make repeated use of the fact that $\partial_\tau \overline{q} = 0$.

Refering back to (8.8), we see that

$$|d_{0,0}|^2 = \frac{1}{2}(q\overline{q})^{1/24}\prod_{n=0}^{\infty}(1-q^n)\prod_{n=0}^{\infty}(1-\overline{q}^n).$$

Now we see that

$$\partial_\tau |d_{0,0}|^2 = 2\pi i |d_{0,0}|^2\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = 0. \qquad (8.17)$$

From equation (8.9), we get:

$$|d_{0,\frac{1}{2}}|^2 = \frac{1}{2}(q\overline{q})^{1/24}\prod_{n=0}^{\infty}(1+q^n)\prod_{n=0}^{\infty}(1+\overline{q}^n),$$

so

$$\partial_\tau |d_{0,\frac{1}{2}}|^2 = 2\pi i |d_{0,\frac{1}{2}}|^2\left(\frac{1}{24} + \sum_{n=1}^{\infty}\frac{nq^n}{1+q^n}\right). \qquad (8.18)$$

(8.10) leads to:

$$|d_{\frac{1}{2},0}|^2 = (q\overline{q})^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})\prod_{n=1}^{\infty}(1-\overline{q}^{n-\frac{1}{2}}).$$

Taking the derivative:

$$\partial_\tau |d_{\frac{1}{2},0}|^2 = -2\pi i |d_{0,\frac{1}{2}}|^2\left(\frac{1}{48} + \sum_{n=1}^{\infty}\frac{(n-\frac{1}{2})q^{n-\frac{1}{2}}}{1-q^{n-\frac{1}{2}}}\right). \qquad (8.19)$$

Finally equation (8.11) gives:

[8] For convenience we will define $Z = 2Z_{\text{Ising}}$, where Z_{Ising} was found in (8.13). However, the resulting vacuum energy will be the same, since $\partial_\tau \ln(2Z) = \partial_\tau \ln(Z)$, and likewise for the anti-holomorphic part.

$$|d_{\frac{1}{2},\frac{1}{2}}|^2 = (q\overline{q})^{-1/48} \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) \prod_{n=1}^{\infty}(1+\overline{q}^{n-\frac{1}{2}}). \tag{8.20}$$

Taking the derivative:

$$\partial_\tau |d_{\frac{1}{2},\frac{1}{2}}|^2 = 2\pi i |d_{0,\frac{1}{2}}|^2 \left(-\frac{1}{48} + \sum_{n=1}^{\infty} \frac{(n-\frac{1}{2})q^n}{1+q^{n-\frac{1}{2}}} \right). \tag{8.21}$$

We get the anti-holomorphic derivatives for each sector simply by exchanging all qs with $\overline{q}$s and making an overall sign change.[9]

Now we can plug the results of of equations (8.17), (8.18), (8.19) and (8.21) and their anti-holomorphic counterparts into our expression for the vacuum energy (8.16). Since $d_{0,0} = 0$, we can neglect the term from equation (8.17). The result is:

$$\begin{aligned} E_0 \;\; = \;\; & \frac{2\pi}{|\theta_2|+|\theta_3|+|\theta_4|} \left(\left[\frac{1}{12} + \sum_{n=1}^{\infty} \left(\frac{nq^n}{1+q^n} + \frac{n\overline{q}^n}{1+\overline{q}^n} \right) \right] |\theta_2(0|\tau)| \right. \\ & - \left[\frac{1}{24} + \sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) \left(\frac{q^{n-\frac{1}{2}}}{1-q^{n-\frac{1}{2}}} + \frac{\overline{q}^{n-\frac{1}{2}}}{1-\overline{q}^{n-\frac{1}{2}}} \right) \right] |\theta_4(0|\tau)| \\ & \left. + \left[-\frac{1}{24} + \sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) \left(\frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} + \frac{\overline{q}^{n-\frac{1}{2}}}{1+\overline{q}^{n-\frac{1}{2}}} \right) \right] |\theta_3(0|\tau)| \right) \end{aligned} \tag{8.22}$$

where we have expressed Z and the $d_{v,u}$ in terms of θ functions, and an overall factor of $|\eta(\tau)|^{-1}$ in the derivatives have been cancelled by a factor $\eta(\tau)$ from the denominator in (8.16).

To understand the meaning of this function we wish to examine its value in the cylindrical limit, obtained by letting $\Im\tau \to \infty$. We then need the limiting values of the θ functions given in (V.19).

Using this along with the observation that all the terms of all the sums in equation (8.22) go to zero in this limit, we see that:

$$E_0^{\tau\to\infty} = \frac{\pi}{6(1+1)} \left(-\frac{1}{2} - \frac{1}{2} \right) = -\frac{\pi}{12}. \tag{8.23}$$

The result is identical to the one found in equation (7.14) when we take into account that our torus had $\omega_1 = 1$, that is, the circumference of the cylinder is 1.

Let us dwell for a moment on the physical interpretation of the behaviour in this limit. We have previously observed that the $(0,0)$ sector

[9] The overall sign change originates from the fact that $\partial_\tau q = 2\pi i q$ whereas $\partial_{\overline{\tau}}\overline{q} = -2\pi i\overline{q}$.

does not contribute to the torus partition function and vacuum energy. This is as should be expected, since these are everywhere periodic fermions. To make them fermions, we had to insert anti-commutation by hand, and this effectively cancels out the fields from this sector.

We now observe that in the cylindrical limit the $(0, 1/2)$ sector no longer contributes to the vacuum energy. Again, this should not be unexpected. In moving to the cylindrical limit, we let $\Im\omega_2 \to \infty$, so in effect, we cut the torus along the ω_1 boundary circle, making a cylinder that we then stretched out to infinite length. Having an anti-commuting boundary condition along ω_2 then has no effect, so we expect the sectors with equal ω_1 boundary conditions to merge together. We see this merging of the $(1/2, u)$ sectors directly from their contribution of $1/2$ each to the vacuum energy. Also, the anti-commutation put in by hand in the ω_1 direction results in cancellation, and so it is as expected that the $(0, 1/2)$ sector should not contribute.

To dissect the vacuum energy still further, we would like to have a closer look at the term in front of $|\theta_2(0|\tau)|$. As we can see from equation (V.35), this therm is the sum of the $E_2^T(z)$ and its antiholomorphic counterpart. Since E_2^T is a tensor under the submodular group Γ_T at level 2, and $|\theta_2(0|\tau)|$ is a scalar under this same group, the product of these two resembles a tensor under this submodular group.[10]

After making this observation, it is tempting to guess that the terms in front of $|\theta_3(0|\tau)|$ and $|\theta_4(0|\tau)|$ are $E_2^S(z) + \overline{E_2^S(z)}$ and $E_2^W(z) + \overline{E_2^W(z)}$ respectively. Though verifying this has not been within the scope of this thesis, one hopes that it might be verified or falsified in the future. If it turns out to be true, the vacuum energy resembles a 2-tensor under the full modular group, because of the transformation properties of the E_2^T, E_2^S and E_2^W.

8.3 The Massive Theory on the Torus

When calculating the partition function in the general massive case, the algebraic method of the previous section is useless, because the Virasoro generators are not meaningful operators when the conformal symmetry has broken down. We must therefore use a different approach.

The calculation given here may be found in (Saluer and Itzykson, 1987). We restate it here because of its nice use of heat kernel methods, and also for the convenience of having the definitions ready for our vacuum energy calculation. A similar calculation of the partition function for the conformal free boson on the torus can be found in (Di Francesco et al., 1997).

[10] By inserting appropriate factors of η, all the factors in the product are invariant, except th facors in front of each θ function. These are presumably E^T, E^S and E^W, and they now carry all the transformation properties of the 2-form E_0.

The partition function can be written in terms of:

$$D_{v,u}^{-2}(\tau, m) = \int D\phi\overline{\phi} \exp\left(- \int_{T^2} d^2x\overline{\phi}(-\Delta + m^2)\phi \right) = \prod_{n_1,n_2} (E_{n_1n_2})^{-1} \tag{8.24}$$

where Δ is the Laplacian and $E_{n_1n_2}$ are the eigenvalues of the operator $-\Delta + m^2$:

$$E_{n_1n_2} = \left(\frac{2\pi}{A} \right)^2 |(n_1 + v)\omega_1 + (n_2 + u)\omega_2|^2 + m^2, \tag{8.25}$$

$n_1, n_2 \in \mathbb{Z}$ and $A = |\omega_1|^2 \Im\tau$ is the area of he torus. Introducing

$$G(s) = \sum_{n_1,n_2} (E_{n_1n_2})^{-s} \tag{8.26}$$

we see that

$$D_{u,v}^{-2} = \prod_{n_1,n_2} e^{-\ln E_{n_1n_2}} = e^{-\sum_{n_1n_2} \ln E_{n_1n_2}}$$

$$G'(0) = -\sum_{n_1n_2} \ln E_{n_1n_2}.$$

We can write this as

$$D_{u,v} = e^{-\frac{1}{2}G'(s)}. \tag{8.27}$$

Defining $t = m|\omega_1|/(2\pi)$ a dimensionless scaling quantity and factorizing $G(s)$ into

$$G(s) = \left(\frac{A\Im\tau}{4\pi^2} \right)^s g(s),$$

where

$$g(s) = \sum_{n_1} \sum_{n_2} \left((n_1 + v + (n_2 + u)\Re\tau)^2 + \Im\tau^2 \left((n_2 + u)^2 + t^2 \right) \right)^{-s} \tag{8.28}$$

we find that

$$D_{v,u} = e^{\frac{1}{2}g(0) \ln \frac{A\Im\tau}{2\pi} + \frac{1}{2}g'(0)}. \tag{8.29}$$

To make writing less tedious we define

$$a = v + (n_2 + u)\Re\tau, \qquad b = \Im\tau \left((n_2 + u)^2 + t^2 \right)^{1/2} > 0. \tag{8.30}$$

We can now partially perform the summation over n_1 by making a Fourier expansion:

$$\sum_{n_1} \left((n_1 + a)^2 + b^2 \right)^{-s} = \sum_{n_1} e^{2\pi i n_1 a} \int_{-\infty}^{+\infty} dy e^{-2\pi i n_1 y} \left(y^2 + b^2 \right)^{-s} \tag{8.31}$$

Making use of the definition of the Γ function (III.1), we rewrite the sum in (8.31) as:

$$\sum_{n_1} e^{2\pi i n_1 a} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dx}{x} x^s \int_{-\infty}^{+\infty} dy e^{-x(y^2+b^2)-2\pi i n_1 y} \tag{8.32}$$

Performing the Gaussian integral over y in equation (8.32), yields:

$$\frac{\sqrt{\pi}}{\Gamma(s)} \sum_{n_1} e^{2\pi i n_1 a} \int_0^\infty \frac{dx}{x} x^{s-1/2} e^{-(xb^2+\frac{n_1^2\pi^2}{x})} \tag{8.33}$$

Rescaling the integration and taking out the $n_1 = 0$ term from the sum (denoting a sum over all $n \neq 0$ by $\sum_n'$), what we get is:

$$\begin{aligned} \sum_{n_1} \left((n_1+a)^2+b^2\right)^{-s} &= \frac{\sqrt{\pi}}{\Gamma(s)} \left[\frac{\Gamma(s-1/2)}{b^{2s-1}} + \sum_{n_1}{}' e^{2\pi i n_1 a} \left(\frac{\pi|n_1|}{b}\right)^{s-1/2} \right. \\ & \left. \times \int_0^\infty \frac{dy}{y} y^{s-1/2} e^{(-\pi|n_1|b(y+1/y))} \right] \end{aligned}$$

Letting $s \to 0$, we will now make use of the fact that:

$$\int_0^\infty dy y^{(\pm 1/2-1)} e^{-\lambda(y+1/y)} = \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-2\lambda} \tag{8.34}$$

We reinsert the values of a and b and the sum over n_2 to find the value of $g(s)$:[11]

$$\begin{aligned} g(s) &= \frac{\sqrt{\pi}}{\Gamma(s)} \Gamma(s-\frac{1}{2}) \Im\tau^{1-2s} \sum_n \left((n+u)^2+t^2\right)^{\frac{1}{2}-s} \\ & -2s \ln \left| \prod_n \left[1 - e^{2\pi i \left(v + \left(\frac{n}{2}+u\right)\Re\tau\right) - 2\pi\Im\tau\left(\left(\frac{n}{2}+u\right)^2+t^2\right)^{1/2}} \right] \right| + O(s^2) \end{aligned} \tag{8.35}$$

We observe that the first term is independent of v, and we give this first term a name of its own:

$$\hat{g}_u(s) = \frac{\sqrt{\pi}}{\Gamma(s)} \Gamma(s-\frac{1}{2}) \Im\tau^{1-2s} \sum_{-\infty}^{\infty} \left((n+u)^2+t^2\right)^{\frac{1}{2}-s} \tag{8.36}$$

[11] In general we imagine that the boundary conditions u and v can be any number of the form $v = k/N, u = l/N, k, l, N \in \mathbb{N}, k, l < N$. If such general conditions are used, most of our calculation will be identical, however the occurences of $(n/2+u)$ must be swapped with $(n+l)/N$ if $u = l/N$. For instance, a Q-state Potts model has $N = Q$ and $k, l = 0, 1, \ldots, Q-1$. In the final expressions we obtain, however, this distinction will have vanished, and a simple substitution for v, u will be valid for all v, u of the type discussed here.

To be able to compare the vacuum energy with the cylinder expression (7.15), we want to express $\hat{g}_u(s)$ as an expansion in t. We will do this by taking out the first few terms, and collecting the higher order terms in one part. In addition to this, we to pull out the s dependence so that taking the $s \to 0$ limit is simplified.

For convinience, we pull an extra factor $\sqrt{\pi}$ wheras we multiply each term with the common factor of $\Gamma(s-1/2)$. Next we rewrite the rest of the sum like this:

$$\begin{aligned} \sum_n \left((n+u)^2+t^2\right)^{\frac{1}{2}-s} &= \sum_n (n+u)^{1-2s}\left(1-\frac{(s-\frac{1}{2})t^2}{(n+u)^2}\right) \\ &\quad +(s-\frac{1}{2})(s+\frac{1}{2})t^4\int_0^1 d\lambda \sum_n \frac{1-\lambda}{(n^2+\lambda t^2)^{s+3/2}} \end{aligned} \tag{8.37}$$

The cases $u=0$, $u=1/2$ are of special interest to us. Consider first the case of $u=0$. In this case we would like to pull out the $n=0$ term. This yields one single term of order t^{1-2s}. Then the rest of the sum in equation (8.37) is handled by letting each $\sum_n \to 2\sum_{n=1}^{\infty}$. The result for $\hat{g}_0(s)$ is then:[12]

$$\begin{aligned} \hat{g}_0(s) &= \frac{\pi\Im\tau^{1-2s}}{\Gamma(s)}\Bigg[\frac{t^{1-2s}\Gamma(s-\frac{1}{2})}{\sqrt{\pi}}+\frac{2\Gamma(s-1/2)\zeta(2s-1)}{\sqrt{\pi}} \\ &\quad -\frac{2\Gamma(s+1/2)\zeta(2s+1)t^2}{\sqrt{\pi}}+\frac{2\Gamma(s+3/2)t^4}{\sqrt{\pi}}\int_0^1 d\lambda\sum_{n=1}^{\infty}\frac{1-\lambda}{(n^2+\lambda t^2)^{s+3/2}}\Bigg] \end{aligned} \tag{8.38}$$

Taking into account that

$$\zeta(-1) = -1/12, \qquad \zeta(1+2s) = 1/2s(1+2s\gamma+\ldots)$$
$$\Gamma(s+1/2) = \sqrt{\pi}[1-s(\gamma+2\ln 2)+\ldots]$$
$$\Gamma(-1/2) = -2\sqrt{\pi}, \qquad 1/\Gamma(s) = s(1+\gamma s+\ldots)$$

where γ is Eulers constant as defined in equation (III.6):

$$\begin{aligned} \hat{g}_0(s) &= \pi\Im\tau\Bigg[-t^2+s\Bigg(\frac{1}{3}-2t+2t^2\ln\left(2\Im\tau e^{-\gamma}\right) \\ &\quad +t^4\int_0^1 d\lambda\sum_{n=1}^{\infty}\frac{1-\lambda}{(n^2+\lambda t^2)^{3/2}}\Bigg)\Bigg]+O(s^2) \end{aligned} \tag{8.39}$$

[12]We have made use of the definition of the Riemann ζ function, here denoted simply ζ, found in (III.18). The recursion relation for the Γ function, to be be found in equation (III.2), has also been used a couple of times.

For general u, we find:

$$
\begin{aligned}
\hat{g}_u(s) &= \frac{\pi\Im\tau^{1-2s}}{\Gamma(s)}\left[\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}}\sum_n \frac{1}{|n+u|^{2s-1}}\right. \\
&\quad -\frac{\Gamma(s+1/2)t^2}{\sqrt{\pi}}\sum_n \frac{1}{|n+u|^{2s+1}} \\
&\quad \left. +\frac{\Gamma(s+3/2)t^4}{\sqrt{\pi}}\int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+u)^2+\lambda t^2\right)^{s+3/2}}\right] \qquad (8.40)
\end{aligned}
$$

Taking the limit when $s \to 0$ in the first term yields:

$$
\lim_{s\to 0}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}}\sum_n \frac{1}{|n+u|^{2s-1}} = \frac{1}{3}\left[1-6u\,(1-u)\right]
$$

The value $\Gamma(3/2) = \sqrt{\pi}/2$ can simply be inserted in front of the last term, so that:

$$
\begin{aligned}
\hat{g}_u(s) &= \pi\Im\tau s\left(\frac{1}{3}\left[1-6u\,(1-u)\right]\right. \\
&\quad \left. +\frac{t^4}{2}\int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+u)^2+\lambda t^2\right)^{s+3/2}}\right) \\
&\quad -\pi\Im\tau t^2 \frac{\Im\tau^{-2s}\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)}\sum_n \frac{1}{|n+u|^{2s+1}} + O(s^2). \qquad (8.41)
\end{aligned}
$$

Considering the last term of (8.41):

$$
\frac{\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)}\sum_n \frac{1}{|n+u|^{2s+1}} = \frac{\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s)}\left(H\,(u;1+2s)+H\,(1-u;1+2s)\right) \qquad (8.42)
$$

where

$$
H(x;\sigma) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^\sigma} \qquad (8.43)
$$

$$
\lim_{\sigma\to 1} H(x;\sigma) \simeq \frac{1}{\sigma-1} + \alpha(x) + \ldots. \qquad (8.44)
$$

We can determine $\alpha(x)$ from the following observations:

$$
\frac{\partial}{\partial x}H(x;\sigma) = -\sigma H(x;\sigma+1) \qquad (8.45)
$$

$$
\frac{\partial^2}{\partial x\partial\sigma}H(x;\sigma) = -H(x;\sigma+1) - \sigma\frac{\partial}{\partial\sigma}H(x;\sigma+1). \qquad (8.46)
$$

Performing the σ derivation in equation (8.46) on equation (8.44) and inserting for H, we realise that:

$$\lim_{\sigma\to 0} \frac{\partial^2}{\partial x \partial \sigma} H(x;\sigma) = -\alpha(x) + \dots,$$

and since

$$\frac{\partial}{\partial \sigma} H(x;\sigma)|_{\sigma=0} = \ln \frac{\Gamma(x)}{(2\pi)^{1/2}},$$

we realise that $\alpha(x)$ is related to the digamma function $\psi(x)$ defined in equation (III.8)

$$\alpha(x) = -\psi(x). \tag{8.47}$$

Now

$$\begin{aligned} \hat{g}_u(s) &= \pi \Im\tau \Bigg(-t^2 + s \Bigg[\frac{1}{3} \left(1 - 6u\left(1-u\right)\right) \\ &\quad + t^2 \left(2 \ln \Im\tau + \psi(u) + \psi(1-u) + 2\ln 2\right) \\ &\quad + \frac{t^4}{2} \int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+u)^2 + \lambda t^2\right)^{3/2}} \Bigg] \Bigg) + O(s^2). \end{aligned} \tag{8.48}$$

Returning to the full $D_{v,u}$- function

$$\begin{aligned} D_{v,u}(\tau, m) &= \exp(-\pi \Im\tau \gamma_u) \Bigg| \prod_n \Bigg[1 - \exp\Bigg(2\pi i \left(v + (n+u)\Re\tau\right) \\ &\quad - 2\pi \Im\tau \left((n+u)^2 + t^2\right)^{1/2} \Bigg) \Bigg] \Bigg|, \end{aligned} \tag{8.49}$$

where

$$\gamma_0 = \frac{1}{6} - t + t^2 \ln \left(4\pi e^{-\gamma} \left(\frac{\Im\tau}{A} \right)^{1/2} \right) + \frac{t^4}{2} \int_0^1 d\lambda \sum_n \frac{1-\lambda}{(n^2 + \lambda t^2)^{3/2}}$$

$$\begin{aligned} \gamma_{u(u\neq 0)} &= \frac{1}{6}\left[1 - 6u\left(1-u\right)\right] + t^2 \left[\ln 4\pi \left(\frac{\Im\tau}{A} \right)^{1/2} + \frac{1}{2}\psi(u) + \frac{1}{2}\psi(1-u) \right] \\ &\quad + \frac{t^4}{2} \int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+u)^2 + \lambda t^2\right)^{3/2}}. \end{aligned} \tag{8.50}$$

Specifically

$$\begin{aligned}\gamma_{\frac{1}{2}} &= -\frac{1}{12} + t^2 \ln \left[\pi e^{-\gamma}\left(\frac{\Im\tau}{A}\right)^{1/2}\right] \\ &+\frac{t^4}{2}\int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+1/2)^2+\lambda t^2\right)^{s+3/2}}\end{aligned} \tag{8.51}$$

In the previous section we saw that each sector contributed an equal amount to the partition function. We also needed a normalization of $1/2$ which is in fact unessescary in this formulation.

$$Z^{m\neq 0}_{\text{Ising}} = D_{1/2,1/2}(m) + D_{0,1/2}(m) + D_{1/2,0}(m) + D_{0,0}(m), \tag{8.52}$$

so that

$$\begin{aligned}Z^{m\neq 0}_{\text{Ising}} &= \exp \pi \Im\tau \left[\frac{1}{12} - t^2 \ln\left(\pi e^{-\gamma}\left(\frac{\Im\tau}{A}\right)^{1/2}\right)\right. \\ &\left. -\frac{t^4}{2}\int_0^1 d\lambda \sum_{n=1}^{\infty} \frac{1-\lambda}{((n-1/2)^2+\lambda t^2)^{3/2}}\right] \\ &\times \left[\sum_{\pm}\prod_n \left(1 \pm \exp\left[2\pi i (n+\frac{1}{2})\Re\tau\right.\right.\right. \\ &\left.\left. -2\pi\Im\tau\left(\left(n+\frac{1}{2}\right)^2 + t^2\right)^{1/2}\right]\right) \\ &+ \exp -\pi\Im\tau\left[\frac{1}{4} - t + t^2\ln 4 - \frac{t^4}{2}\int_0^1 d\lambda(1-\lambda)\right. \\ &\left.\times \sum_n \frac{1}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}} - \frac{1}{(n^2+\lambda t^2)^{3/2}}\right] \\ &\left.\times \sum_{\pm}\prod_n\left(1\pm\exp\left(2\pi i n \Re\tau - 2\pi\Im\tau\left(n^2+t^2\right)^{1/2}\right)\right)\right].\end{aligned} \tag{8.53}$$

Looking at how the terms in equation (8.53) came about from the different boundary condition parts, we see that the first sum over $\pm$ gives the sum of $D_{1/2,1/2}(m)$ and $D_{0,1/2}(m)$, with the $+$ part for the former and the $-$ part for the latter. The second sum over $\pm$ then gives $D_{1/2,0}(m) + D_{0,0}(m)$, again with the $+$ part for the former and the $-$ part for the latter.

In the limit $m \to 0$, $t \to 0$, $D_{v,u}(0) = |d_{v,u}|^2$, so by comparing the (8.53) in the massless limit to the equations (8.12) and (8.13), we see that we have obtained the proper conformal limit.

8.3.1 Vacuum energy

We now apply the expression in equation (8.15) for the vaccuum energy to our massive partition function (8.53). However, since our expression is in fact given directly in terms of $\Im\tau$ and $\Re\tau$, we will use the expression in terms of $\partial_{\Im\tau}$. We start by introducing some convenient notation:

$$\begin{aligned}
\varphi &= \pi\left[\frac{1}{12} - t^2 \ln\left(\pi e^{-\gamma}\left(\frac{\Im\tau}{A}\right)^{1/2}\right)\right. \\
&\quad \left. -\frac{t^4}{2}\int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}}\right] \qquad (8.54) \\
\epsilon &= -\pi\left[\frac{1}{4} - t + t^2\ln 4 - \frac{t^4}{2}\int_0^1 d\lambda(1-\lambda)\right. \\
&\quad \left. \times\sum_n \frac{1}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}} - \frac{1}{(n^2+\lambda t^2)^{3/2}}\right] \qquad (8.55) \\
\xi_1 &= \sum_\pm \prod_n \left(1 \pm \exp\left[2\pi i\left(n+\frac{1}{2}\right)\Re\tau\right.\right. \\
&\quad \left.\left. -2\pi\Im\tau\left(\left(n+\frac{1}{2}\right)^2+t^2\right)^{1/2}\right]\right) \qquad (8.56) \\
\xi_2 &= \sum_\pm \prod_n \left(1 \pm \exp\left[2\pi i n\Re\tau - 2\pi\Im\tau\left(n^2+t^2\right)^{1/2}\right]\right) \qquad (8.57) \\
\xi &= \xi_1 + \exp(\Im\tau\epsilon)\xi_2 \qquad (8.58)
\end{aligned}$$

In summary:

$$Z_{\text{Ising}}^{m\neq 0} = \exp(\Im\tau\varphi)\left(\xi_1 + \exp\left(\Im\tau\epsilon\right)\xi_2\right) = \exp(\Im\tau\varphi)\xi, \qquad (8.59)$$

so we can write:

$$E_0^{m\neq 0} = -\frac{\partial lnZ}{\partial\Im\tau} = -\frac{\partial\Im\tau\varphi}{\partial\Im\tau} - \frac{1}{\xi}\frac{\partial\xi}{\partial\Im\tau}. \qquad (8.60)$$

To simplyfy our calculation we will assume $\omega_1 = 1$, which is no loss of generality since we can always convert to a system of units where this is true. In the result, we may then simply convert back to suitable units.

We start with A a constant, then after taking the derivative we plug in the knowledge that $A = |\omega_1|^2\Im\tau$ and that we can set $\omega_1 = 1$ so that $\Im\tau/A = 1$. Hence the derivation of φ yields:

$$-\frac{\partial\Im\tau\varphi}{\partial\Im\tau} = -\varphi + \frac{1}{2}\pi t^2. \qquad (8.61)$$

Looking at the derivative of ξ, we first divide it into two parts:

$$-\frac{1}{\xi}\frac{\partial\xi}{\partial\Im\tau} = -\frac{1}{\xi}\left(\frac{\partial\xi_1}{\partial\Im\tau} + e^{\Im\tau\epsilon}\frac{\partial\xi_2}{\partial\Im\tau} + \epsilon e^{\Im\tau\epsilon}\xi_2\right) \tag{8.62}$$

We continue by finding the derivatives of ξ_1 and ξ_2:

$$\begin{aligned}\frac{\partial\xi_1}{\partial\Im\tau} &= \sum_{\pm}\prod_{n}\left(1 \pm \exp\left[2\pi i(n+\frac{1}{2})\Re\tau - 2\pi\Im\tau\left((n+\frac{1}{2})^2+t^2\right)^{1/2}\right]\right)\\ &\quad\times\sum_{k}\frac{\mp 2\pi\left((k+\frac{1}{2})^2+t^2\right)^{1/2}}{1\pm\exp\left[2\pi i(k+\frac{1}{2})\Re\tau - 2\pi\Im\tau\left((k+\frac{1}{2})^2+t^2\right)^{1/2}\right]}\end{aligned} \tag{8.63}$$

$$\begin{aligned}\frac{\partial\xi_2}{\partial\Im\tau} &= \sum_{\pm}\prod_{n}\left(1 \pm \exp\left[2\pi in\Re\tau - 2\pi\Im\tau(n^2+t^2)^{1/2}\right]\right)\\ &\quad\times\sum_{k}\frac{\mp 2\pi(k^2+t^2)^{1/2}}{1\pm\exp\left[2\pi ik\Re\tau - 2\pi\Im\tau(k^2+t^2)^{1/2}\right]}\end{aligned} \tag{8.64}$$

The expression for the vacuum energy then reads:

$$
\begin{aligned}
E_0^{m\neq 0} \quad = \quad & -\pi\Bigg[\frac{1}{12}-t^2\ln\left(\pi e^{-\gamma+1/2}\left(\frac{\Im\tau}{A}\right)^{1/2}\right) \\
& -\frac{t^4}{2}\int_0^1 d\lambda\sum_n\frac{1-\lambda}{\left((n+1/2)^2+\lambda t^2\right)^{s+3/2}}\Bigg] \\
& -\Bigg[\sum_{\pm}\prod_n\left(1\pm\exp\left[2\pi i(n+\frac{1}{2})\Re\tau-2\pi\Im\tau\{(n+\frac{1}{2})^2+t^2\}^{1/2}\right]\right) \\
& \times\sum_k\frac{\mp 2\pi\left(\left(k+\frac{1}{2}\right)^2+t^2\right)^{1/2}}{1\pm\exp\left[2\pi i(k+\frac{1}{2})\Re\tau-2\pi\Im\tau\left(\left(k+\frac{1}{2}\right)^2+t^2\right)^{1/2}\right]} \\
& +\exp-\pi\Im\tau\Bigg(\frac{1}{4}-t+t^2\ln 4-\frac{t^4}{2}\int_0^1 d\lambda(1-\lambda) \\
& \times\sum_n\frac{1}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}}-\frac{1}{(n^2+\lambda t^2)^{3/2}}\Bigg) \\
& \times\Bigg(-\pi\Bigg\{\frac{1}{4}-t+t^2\ln 4-\frac{t^4}{2}\int_0^1 d\lambda(1-\lambda) \\
& \times\sum_n\frac{1}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}}-\frac{1}{(n^2+\lambda t^2)^{3/2}}\Bigg\} \\
& \times\sum_{\pm}\prod_n\left(1\pm\exp\left[2\pi in\Re\tau-2\pi\Im\tau(n^2+t^2)^{1/2}\right]\right) \\
& +\sum_{\pm}\prod_n\left(1\pm\exp\left[2\pi in\Re\tau-2\pi\Im\tau(n^2+t^2)^{1/2}\right]\right) \\
& \times\sum_k\frac{\mp 2\pi(n^2+t^2)^{1/2}}{1\pm\exp\left[2\pi ik\Re\tau-2\pi\Im\tau(k^2+t^2)^{1/2}\right]}\Bigg)\Bigg] \\
& \times\Bigg[\sum_{\pm}\prod_n\left(1\pm\exp\left[2\pi i(n+\frac{1}{2})\Re\tau-2\pi\Im\tau\{(n+\frac{1}{2})^2+t^2\}^{1/2}\right]\right) \\
& +\exp-\pi\Im\tau\Bigg(\frac{1}{4}-t+t^2\ln 4-\frac{t^4}{2}\int_0^1 d\lambda(1-\lambda) \\
& \times\sum_n\frac{1}{\left((n+1/2)^2+\lambda t^2\right)^{3/2}}-\frac{1}{(n^2+\lambda t^2)^{3/2}}\Bigg) \\
& \times\sum_{\pm}\prod_n\left(1\pm\exp\left[2\pi in\Re\tau-2\pi\Im\tau(n^2+t^2)^{1/2}\right]\right)\Bigg]^{-1}
\end{aligned}
\tag{8.65}
$$

To get some information out of this complicated expression, we need to dissect it slightly.

A closer look at the different parts in the cylinder limit (i.e. when $\Im\tau \to \infty$ while $\Re\tau$ remains constant) reveals that only φ will give a non-zero limit, namely:

$$
\begin{aligned}
E_0^{\Im\tau\to\infty} &= -\pi\left[\frac{1}{12} - \left(\frac{m}{2\pi}\right)^2 \ln\left(\pi e^{-\gamma+1/2}\right)\right. \\
&\quad \left. -\frac{1}{2}\left(\frac{m}{2\pi}\right)^4 \int_0^1 d\lambda \sum_n \frac{1-\lambda}{\left((n+1/2)^2 + \lambda\left(\frac{m}{2\pi}\right)^2\right)^{3/2}}\right]
\end{aligned}
\tag{8.66}
$$

for a cylinder of circumference $\beta = 1$.

By comparing (8.66) with the results of (7.14) and (8.23), we see that the conformal limit is the right one.

To see that the same is true for the low mass expansion requires a small argument. To get the expression found in equation (7.15), we need to use the normalisation $A = \pi^2\Im\tau$ rather than $A = \Im\tau$. This has to do with the definitions of the quantities, so is not of essential importance. Doing this and putting $\beta = 1$ in equation (7.15), we see the two expressions (7.15) and (8.66) are equal up to second order.

As for the conformal limit of the full torus vacuum energy, a few lines of writing will show us that it is:

$$
\begin{aligned}
E_0^{m=0} &= -\frac{\pi}{12} + \frac{2\pi}{|\theta_2|+|\theta_3|+|\theta_4|}\left[\left(\frac{1}{8} + \sum_{n=1}^{\infty}\left(\frac{nq^n}{1+q^n} + \frac{n\overline{q}^n}{1+\overline{q}^n}\right)\right)|\theta_2(0|\tau)|\right. \\
&\quad -\left(\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)\left(\frac{q^{n-\frac{1}{2}}}{1-q^{n-\frac{1}{2}}} + \frac{\overline{q}^{n-\frac{1}{2}}}{1-\overline{q}^{n-\frac{1}{2}}}\right)\right)|\theta_4(0|\tau)| \\
&\quad \left.+\left(\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)\left(\frac{q^{n-\frac{1}{2}}}{1+q^{n-\frac{1}{2}}} + \frac{\overline{q}^{n-\frac{1}{2}}}{1+\overline{q}^{n-\frac{1}{2}}}\right)\right)|\theta_3(0|\tau)|\right]
\end{aligned}
\tag{8.67}
$$

which is what we found in equation (8.22) in section 8.2.1.

Chapter 9

The Ising BCFT and its Connection to the Massive Field Theory

> Ved tingenes ytterste grense
> finnes det ingen regler

Seng-ts'an *Hsin-hsin-ming* or *Of the faithful mind* as given in (Thelle, 2001).

In this chapter we will explore the interesting rules imposed by the presence of a boundary. The most interesting result will be a peculiar equality between non-critical theories with integrability and massive theories with free boundary conditions, theories where there in a sense are no rules at the outer boundary.

Cardy's les Houches lectures of 1988 (Cardy, 1990) were crowned by the exploration of the very interesting fact that the eigenvalue spectrum of the corner transfer matrix of integrable models are generated by a Virasoro algebra.

This fact had then already been proved in (Date et al., 1989, 1987, 1986) and (Itoyama and Thacker, 1987). Later the possible reasons for this similarity was treated in (Itoyama and Thacker, 1989) and (Saluer and Bauer, 1989).

Below, we will first explore the BCFT of the Ising model, then we will look at the non-critical model in terms of the corner transfer matrix, and finally we will try to convey where the similarity may come from, as explained by (Itoyama and Thacker, 1989) and (Saluer and Bauer, 1989).

Although we do this here only for the Ising model, the relations that we find between non-critical models and corresponding BCFTs appears to be true for all integrable models (Itoyama and Thacker, 1989; Saluer and Bauer, 1989). Thus is a very interesting property of integrable theories as such, an may shed considerable light on the nature of integrability.

9.1 The Ising BCFT

We first compute the partition functions of the Ising model on an annulus for different boundary conditions. The derivation is based on similar derivations found both in (Cardy, 2008) and (Di Francesco et al., 1997).

Recall from section 4.3.5 that we only need the modular matrix $S_{h'h}$ to find the boundary operators corresponding to the different boundary conditions.

From (Di Francesco et al., 1997) and (Cardy, 2008) we find the modular matrix to be:[1]

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Applying this to equations (4.19) and (4.20) we read off that:

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\frac{1}{2}\rangle + \frac{1}{2^{1/4}}|\frac{1}{16}\rangle \tag{9.1}$$

$$|\tilde{\frac{1}{2}}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\frac{1}{2}\rangle - \frac{1}{2^{1/4}}|\frac{1}{16}\rangle \tag{9.2}$$

$$|\tilde{\frac{1}{16}}\rangle = |0\rangle - |\frac{1}{2}\rangle \tag{9.3}$$

We notice that the first two boundary states, (9.1) and (9.2) are the same except for a sign change in front of the $|1/16\rangle$ state. Since $|1/16\rangle$ is the spin state of the theory, we can identify these boundary conditions with the $+$ and $-$ boundary conditions. The last state, equation (9.3), corresponds to the free boundary condition.

The fusion algebra of the Verma modules $\mathcal{V}_h$ is given by:[2]

$$\mathcal{V}_{\frac{1}{16}} \odot \mathcal{V}_{\frac{1}{16}} = \mathcal{V}_0 + \mathcal{V}_{\frac{1}{2}} \tag{9.4}$$

$$\mathcal{V}_{\frac{1}{16}} \odot \mathcal{V}_{\frac{1}{2}} = \mathcal{V}_{\frac{1}{16}} \tag{9.5}$$

$$\mathcal{V}_{\frac{1}{2}} \odot \mathcal{V}_{\frac{1}{2}} = \mathcal{V}_0 \tag{9.6}$$

From this we find the non-zero boundary operator content $n^h_{\tilde{a}\tilde{b}}$ of a theory with boundary conditions (a, b): $n^h_{\tilde{h}\tilde{0}} = n^0_{\tilde{\frac{1}{16}}\tilde{\frac{1}{16}}} = n^{\frac{1}{2}}_{\tilde{\frac{1}{16}}\tilde{\frac{1}{16}}} = n^0_{\tilde{\frac{1}{2}}\tilde{\frac{1}{2}}} = n^{\frac{1}{16}}_{\tilde{\frac{1}{16}}\tilde{\frac{1}{2}}} = 1$.

Using equation (4.12) we find that the partition functions for the different boundary conditions are:

[1] Making use of formula 10.134 of (Di Francesco et al., 1997), it is not difficult to find this matrix.

[2] Since $|0\rangle$ is the identity operator, the fusion between any Verma module $\mathcal{V}_h$ and $\mathcal{V}_0$ is just $\mathcal{V}_h$.

$$
\begin{aligned}
Z_{++}(\tau) &= \chi_0(q) & Z_{--}(\tau) &= \chi_0(q) \\
Z_{f+}(\tau) &= \chi_{\frac{1}{16}}(q) & Z_{f-}(\tau) &= \chi_{\frac{1}{16}}(q) \\
Z_{-+}(\tau) &= \chi_{\frac{1}{2}}(q) & Z_{ff}(\tau) &= \chi_0(q) + \chi_{\frac{1}{2}}(q) \qquad (9.7)
\end{aligned}
$$

In terms of the sector wise partition functions given in equations (8.8), (8.9), (8.10), and (8.11), the characters are:[3]

$$
\begin{aligned}
\chi_0 &= \frac{1}{2}(d_{\frac{1}{2},\frac{1}{2}} + d_{\frac{1}{2},0}) \\
\chi_{\frac{1}{2}} &= \frac{1}{2}(d_{\frac{1}{2},\frac{1}{2}} - d_{\frac{1}{2},0}) \\
\chi_{\frac{1}{16}} &= \frac{1}{\sqrt{2}} d_{0,\frac{1}{2}} \qquad (9.8)
\end{aligned}
$$

From this we can express the partition functions in terms of Jacobi θ functions and the Dedekind η function as in equation 8.2, and calculations of measurable quantities can be calculated as done in chapter 8.

Notice that $|1/16\rangle$ and $|1/2\rangle$ are so-called boundary changing operators. That is, the input of $|1/16\rangle$ can cause the transition between free and $+$ or $-$ boundary conditions, whereas the insertion of a $|1/2\rangle$ field can change the boundary conditions from $+$ to $-$ (Di Francesco et al., 1997).

It is also interesting to observe that all the partition functions are invariant under the change between $+$ and $-$. This reflects the symmetry between spin up and spin down. Note also that the partition function for free-free boundary conditions equals the sum of the partition functions for fixed equal and fixed unequal boundary conditions.

9.2 The Massive Model Partition Function as Calculated with the Corner Transfer Matrix

In this section, we will describe how to obtain results for a non-critical massive model on the plane using the corner transfer matrix.

We will not give a full derivation of the results and the calculations, since this has been done in (Baxter, 1982) and in a shorter version in (Cardy, 1990). Instead, we will give an account of the main ideas involved and relate the results.

The corner transfer matrix[4] is defined for all interaction round a face (IRF) models. IRF models are models where you can find the entire

[3]For a derivation see for instance (Di Francesco et al., 1997).

[4]Sometimes denoted CTM.

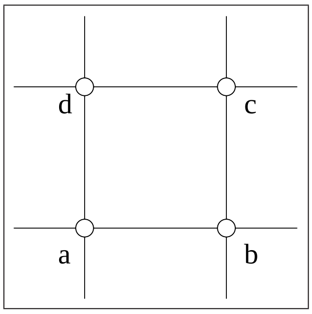

Figure 9.1: The spins a, b, c and d around a face. This face has the Boltzmann weight $w(a, b, c, d)$.

partition function through summing the product of weights around each face, $w(a, b, c, d)$, over the possible spin configurations. a, b, c and d are the spins in the corners of each face, as shown i figure 9.1.

To solve for the Ising model, we normally start with a double Ising lattice, the two lattices having different couplings K_1 and K_2. This means that in figure 9.1, a and c will be coupled with the coupling K_1, and b and d with the coupling K_2.

If the product $\sinh 2K_1 \sinh 2K_2 = k^{-1}$ is the same for all faces, the row-to-row transfer matrices[5] commute, and we can parametrise the Boltzmann weights by elliptic functions in the following way:

$$e^{-2K_1} = \mathrm{snh}u/\mathrm{snh}\lambda \qquad e^{-2K_2} = \mathrm{snh}(\lambda - u)/\mathrm{snh}\lambda, \tag{9.9}$$

where $\mathrm{shn}u \equiv -i\mathrm{sn}iu$, sn is the Jacobi elliptic function as defined in Appendix V.2 and λ is given by $k\mathrm{shn}^2\lambda = 1$.

The corner transfer matrix A is the matrix whose element $A_{\sigma,\sigma'}$ is defined to be the partition function for an $m \times m$ quadrant or corner with boundary values set to the ground state, and the values on the y- and x-axis set to σ and σ'. σ and σ' are configurations of spins $\sigma = \{s_1, s_2 \dots s_m\}$ and $\sigma' = \{s'_1, s'_2 \dots s'_m\}$, as shown i figure 9.2. So:

$$A_\sigma \sigma' = \delta_{s_1, s'_1} \sum_{\text{spins}} \prod_{\text{faces}} w(a, b, c, d) \tag{9.10}$$

where the sum over spins is over all the inner spins.

We define the other three corner transfer matrices B, C, and D to be A rotated by 90°, 180°, and 270° respectively.

Observe that the partition function for the entire plane is:

$$\mathcal{Z} = \mathrm{Tr}ABCD \tag{9.11}$$

[5] For more on these, see (Baxter, 1982).

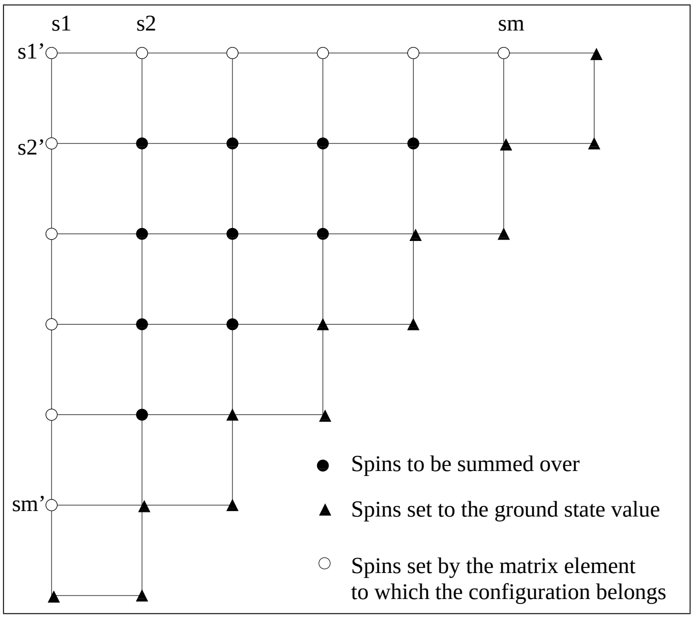

Figure 9.2: The spin configuration for which the element $A_{\sigma,\sigma'}$ of the corner transfer matrix is the partition function. The spins on the boundary are set to the ground state value, for instance they may all point upwards. These spins are denoted by triangles. The white circle spins are defined by the indices of the matrix element to which this configuration belongs. It is obvious from the drawing that $A_{\sigma,\sigma'} = 0$ if $s_1 \neq s_1'$. $A_{\sigma,\sigma'}$ is then the sum over configurations of the black circle spins of the products of weights of all the faces in this drawing.

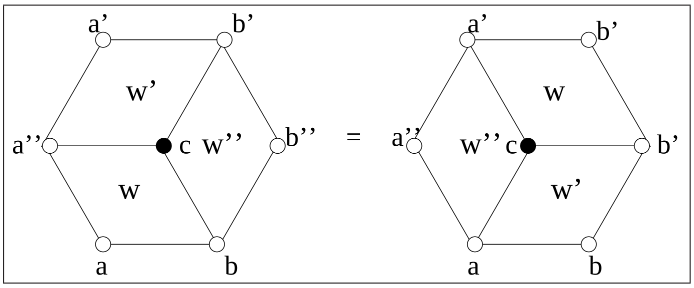

Figure 9.3: Graphical representation of the star-triangle relation as given in equation (9.13). As in figure 9.2 the spins denoted with black circles, c, are summed over.

where A, B, C and D are now taken to their infinite m values.

Notice that in the case of commuting transfer matrices on the Ising model defined by equation (9.9), rotating $90°$ means $u \to \lambda - u$. In this case the parition function is

$$\mathcal{Z} = \mathrm{Tr} A(u)A(\lambda - u)A(u)A(\lambda - u) \tag{9.12}$$

The weights for integrable models obey the star-triangle or Yang-Baxter relations given by:[6]

$$\begin{aligned}\sum_c w(a,b,c,a'')w'(a'',c,b',a')w''(c,b,b'',b') &= \\ \sum_c w''(a'',a,c,a')w'(a,b,b'',c)w''(c,b'',b',a')\end{aligned} \tag{9.13}$$

and presented graphically in figure 9.3.

The star-triangle relation implies commutation of the row-to-row transfer matrices (Cardy, 1990).

It can now be argued that if λ is considered a constant, the corner transfer matrix A must depend exponentially on u:

$$A(u) = e^{-uH_c} \tag{9.14}$$

where H_c is a matrix defined by this relation.

The elliptic parametrisation of the weights can be used to prove that the eigenvalues of H_c are integers times $\pi/(2K)$, where K is the complete elliptic integral of the first kind evaluated at k. Taking $K_2 \to \infty$ means that $\exp(-2K_1) \sim \exp(-(\pi u)/(2K))$.

[6] See for instance (Baxter, 1982).

This also means that all spins on the same K_2-coupled diagonals are equal. It can then be proven that:

$$Z = \mathrm{Tr}e^{-2\lambda H_c} = \sum_{\{s\}} q^{\frac{1}{4}\sum_{j=0}^{\infty}(j+1)(1-s_j s_j+2)} \tag{9.15}$$

where $q = \exp[-2\pi\lambda/K(k)]$.

The even and odd sublattices decouple. We consider only the even sublattice setting $j = 2k$,[7] and factorize Z into a part for the upper half plane and an equal one for the lower half plane. The factor $n_k = (1/2)(1 - s_{2k}s_{2k+2})$ equals one if there is a domain wall[8] between the spins $2k$ and $2k+2$, otherwise it is zero.

Now two cases are possible: The spin s_0 may be the same as the spin at infinity, or it may be the opposite. In the first case, $\sum_{n_k}$ must be even, so the partition function is:

$$Z_{++} = \sum_{\sum_k n_k even} q^{\sum_{k=0}^{\infty}(k+\frac{1}{2})n_k} = \frac{1}{2}\prod_{k=0}^{\infty}(1+q^{k+\frac{1}{2}}) + \frac{1}{2}\prod_{k=0}^{\infty}(1-q^{k+\frac{1}{2}}) \tag{9.16}$$

The partition function for the opposite spin case is the same two factors with a minus sign between them.

Recalling the BCFT partition functions and the values of the characters and partition functions found in equations (9.8) and (9.7) together with (8.8), (8.9), (8.10) and (8.11), and combining them with the definitions of the θ and η functions given in Appendix V.1, we see that Z_{++} and Z_{+-} are identical to those of the BCFT, except for a factor of $q^{-1/48}$.

Note that the full partition function is given by the sum of these two cases, so it is the partition function for the free-free BCFT. But here, intstead of q being given in terms of the size of the annulus, it is given by the infinite plane massive theory's distance from criticality given by $q = \exp[-2\pi\lambda/K(k)]$, where $\lambda/K(k)$ is a parameter giving the distance from criticality.[9]

It is natural to think that theories that are different from the Ising model on a square lattice are different away from criticality. However, close to the critical point there is only one relevant RG-direction, so the theories should be very similar. We therefore expect the partition function in the general case to be given in the shape of the BCFT free-free partition function, but that the parameter showing the distance from criticality will be different.

[7]This is just making the choice of one particular lattice out of the double latticing we had originally.

[8]Spins on each side of a domain wall have opposite values.

[9]It is usual to say that $k-1$ gives the distance from criticality. However, λ is determined by the relation $k\mathrm{shn}^2\lambda = 1$. From this, it can be shown that $\lambda/K(k) \to 0$ as $k \to 1$.

9.3 The Origins of the Similarity Between the Non-Critical and Boundary Theories

We have seen in the sections above the remarkable resemblance between integrable theories off criticality and the same theories at criticality on a surface with boundaries.

Here we will try first to give a simplistic explanation of why this may come to be in the setting of the Ising model. Then we will try to recount the deeper explanations given in (Itoyama and Thacker, 1987, 1989) and (Saluer and Bauer, 1989).

9.3.1 A naive picture

In our review of boundary conformal field theory in section 4.3.5, we started out with the boundary surface of the upper half plane. We will now return to this picture and from there give a motivation why a BCFT and a non-critical theory may be connected.

In (Di Francesco et al., 1997), one of the ways to explain the BCFT on the upper half plane is by way of mirror images. The lower half plane is included to form the entire complex plane, however, the fields on the lower half plane are constrained to be mirror images of those in the upper half plane. As we saw in section 4.3.5, this cuts the Virasoro algebras down to one, and makes the fields on the lower half plane actual complex conjugates of the ones on the upper half plane.

In the case of the massive Majorana fermions, all that changes is that there is a mass term that connects ψ with $\overline{\psi}$ constraining one in terms of the other. In other words, $\overline{\psi}$ is constrained by ψ just like the fields in the lower half plane in the context of mirror images.

Though this argument is very heuristic, we see that the constraint of the boundary condition and the constraint of the mass term are similar.[10]

9.3.2 The deeper explanations

In (Itoyama and Thacker, 1989), it is shown that the reason for the BCFT-like behaviour of an integrable non-critical 2-dimensional theory is that the integrability or Yang-Baxter relations give rise to a corner transfer matrix with very nice properties that hold in the continuum limit. It is shown that the eigenvalue spectrum of its logarithm is given by $\pi/(2K)$.[11]

[10] This argument is just an argument for the Ising model and not one for every completely integrable one. However, the connection between their BCFTs and non-critical theories is the same.

[11] (Itoyama and Thacker, 1987) interpreted the CTM as related to a Lorentz boost operator. However, (Johannesson, 1992) made a more general derivation of the origins of the Virasoro algebra, and proved that the Lorentz boost interpretation is merely a choice among

From this logarithm one may define the Virasoro generators L_n in a fashion similar to the one for the associated conformal theory. However a mass term will impose restricitons on the two Virasoro algebras reducing them to one. This is just what we see for the boundary conditions of a BCFT and in accordance with our naïve picture. The restriction on the Virasoros obviously disappears at the critical point, and we obtain the expected two Virasoro algebras.

(Saluer and Bauer, 1989) make explicit calculations for Q-state Potts models and show that the non-critical theory corresponds to a free-free boundary condition BCFT.[12] Here the BCFT modular parameter $q = \exp(2\pi i\tau)$ must be identified with the nome of elliptic functions with elliptic modulus k, where $k-1$ parametrises the distance from criticality.[13]

(Saluer and Bauer, 1989) also propose that it is always possible to move away from a 2-dimensional CFT point in RG in an integrable direction, and they obtain the partition function for this integrable model from the one for the free-free BCFT. They use this to propose several non-critical partition functions for theories with unknown Boltzmann weights.

If this proposal holds true one could say that CFT symmetry in two dimensions is actually just a special case of integrability, just as the boundaryless surface is a limit of the free-free boundary condition boundary surface.

---

infinitely many possible choices.

[12]The reason for this particular boundary condition is the that it is neccesary for the continuum limiting processes of the CTM to be sensible.

[13]An explanation of what the nome q is, is given in appendix V.2.

Chapter 10

Holography: The Ising Model on the Screen

> However, because of government cuts, we could manage to provide only a two dimensional screen.

Stephen Hawking on not having a 4-dimensional screen for a diagram.

Though a 2-dimensional screen might not be enough to show something 4-dimensional, it might still be enough for something 3-dimensional. In fact, there is a whole range of 3-dimensional systems that can be fully described by their 2-dimensional edges or dual screens. This concept is called holography, and within it, we might find 3-dimensional duals of our 2-dimensional favorite, the Ising model.

In this chapter we describe the basics of holography and the correspondence it makes between AdS_3 (a short introduction to this spacetime was given in section 3.3.) and a 2-dimensional CFT on its boundary.

In (Maldacena, 1998), it was claimed that all dynamics of AdS reduces to previously known conformal field theory. In this work, we will only consider this correspondence in terms of a correspondence of the so-called entanglement or geometric entropy. A description of this concept based on the treatments of the topic in (Holzhey et al., 1994), (Calabrese and Cardy, 2004) and (Ryu and Takayanagi, 2006a), will therefore be given in section 10.1.

In section 10.2, a description of the holographic interpretation of the entanglement is given. We also describe how to find the entanglement entropy of the CFT from calculations in the corresponding AdS space as given in (Ryu and Takayanagi, 2006b,a).

In section 10.3, we investigate the prospects of finding an AdS_3-related geometry, (i.e. the holographic dual) of the non-critical or massive Ising model.

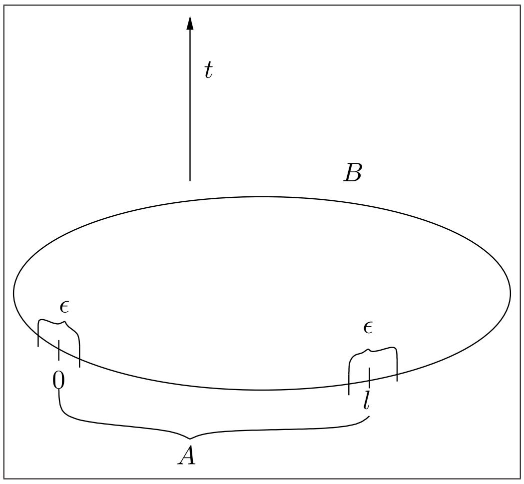

Figure 10.1: The geometry for the entanglement entropy. The circumference of the spatial circle is Λ. The temporal direction is infinite. The observer only has access to the subsystem A of length l.

10.1 Entanglement or Geometric Entropy

As mentioned in chapter 5 (Holzhey et al., 1994) calculated something called entanglement or geometric entropy. They suggested this entropy as a nice candidate for the entropy of a conformal field theory to describe the divergence with ultraviolet cutoff. Their description went roughly as follows.[1]

Start by compactifying the space dimension of a $(1+1)$-dimensional theory. Let it have length Λ and impose periodic boundary conditions. Λ is an infrared cutoff, and the resulting spacetime is a cylinder. Now take out a subsystem A of length l. For convinience, let the space coordinate go from 0 to Λ and let A be the subsystem of spatial coordinate from 0 to l. The rest of the system is then called B (see figure 10.1.).

Let us assume that only subsystem A is accessible to some observer. Now take the entropy to be the von Neumann entropy S_A of A, that is the entropy obtained from the reduced density matrix $\rho_A = \mathrm{Tr}_B\rho$[2]:

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A) \tag{10.1}$$

This entropy is divergent because the conformal field theory has an infintite entropy per volume. Since the boundary between A and B

[1] This description is also inspired by the descriptions given in (Calabrese and Cardy, 2004) and (Ryu and Takayanagi, 2006a).

[2] More on this may be found in (Calabrese and Cardy, 2004), (Ryu and Takayanagi, 2006a) and (Holzhey et al., 1994).

is sharp, the entropy from B along the edge is infinite. However, no instrument has infinite resolution, so a sharp edge is unphysical. A more natural setting is for the boundaries to have a finite length, say ϵ, which is an ultraviolet cutoff.

The entanglement entropy can be found by using the so-called replica trick,[3] and it is shown in (Holzhey et al., 1994) that the result is

$$S = \left(1 - \frac{d}{dn}\right)_{n=1} \ln Z(n), \tag{10.2}$$

where $Z(n)$ is the partition function of the CFT on a torus of cycle lengths $2\pi n/\kappa$ and $2/\kappa \ln(l/\epsilon)$.[4] The result is:

$$S_A = \frac{1}{6}(c + \overline{c}) \ln \frac{l}{\epsilon}, \tag{10.3}$$

under the assumption that $l << \Lambda$. If this is not the case the formula becomes:

$$S_A = \frac{1}{6}(c + \overline{c}) \ln \left(\frac{\Lambda}{\pi\epsilon} \sin \frac{\pi l}{\Lambda} \right). \tag{10.4}$$

Specifically, in the critical Ising model $c = \overline{c} = 1/2$, so

$$S_A^{c=\frac{1}{2}} = \frac{1}{6} \ln \left(\frac{\Lambda}{\pi\epsilon} \sin \frac{\pi l}{\Lambda} \right) \tag{10.5}$$

10.2 Holography - CFT on the Edge

In (Maldacena, 1998) it was first conjectured that *"...all the dynamics of* AdS *reduces to previously known conformal field theories"*. (Witten, 1998) elaborated on this idea and proposed that the correlation functions in the CFT were *"...given by the dependence of the supergravity action on the asymptotic behaviour at infinity. In particular, dimensions of operators in conformal field theory are given by masses of particles in supergravity"*.

Recently there has been much interest in this field, especially since it may offer a way of describing strongly coupled conformal field theories by using weakly coupled gravity.[5] However, our interest here will be its possible description of the Ising model.

In (Ryu and Takayanagi, 2006a,b), a holographic interpretation of the entanglement entropy calculation described in 10.1 is given. Their idea goes roughly as follows.

---

[3]This trick is described both in (Holzhey et al., 1994) and (Calabrese and Cardy, 2004).

[4]κ is an auxiliary parameter with no independent physical meaning and will drop out of all measurable quantities.

[5]For a pedestrian review of the subject and a review of these advances, see (Klebanov and Maldacena, 2009).

First, an AdS_3-space of curvature R corresponds to a CFT with central charge

$$c = \frac{3R}{2G_N^{(3)}} \tag{10.6}$$

where $G_N^{(3)}$ is the 3-dimensional gravitational constant.

Note that the critical Ising model is the holographic dual of a AdS space of curvature $R = G_N^3/3$. It is also interesting to see that the trivial theory, $c = 0$, is the holographic dual of flat AdS-space ($R = 0$).

Using the metric given in equation (3.8), we look at the boundary at some large but finite radius $\rho = \rho_0$. This corresponds to the CFT description above with $\exp(\rho_0) = \Lambda/(\pi\epsilon)$.

Taking a finite timeslice, the entanglement entropy is given in terms of the geodesic distance L_{γ_A} between the two boundary points of A:

$$S_A = \frac{L_{\gamma_A}}{4G_N}. \tag{10.7}$$

This is actually the Beckenstein-Hawking entropy for BPS black holes. Solving the geodesic equations will then give us that:

$$S_A \approx \frac{R}{4G_N^3 \ln\left(e^{2\rho_0}\sin^2\frac{\pi l}{\Lambda}\right)} = \frac{c}{3}\ln\left(\frac{\Lambda}{\pi\epsilon}\sin\frac{\pi l}{\Lambda}\right). \tag{10.8}$$

We see that through the holographic picture, we can find the entanglement entropy by calculating geodesic lengths in the AdS space. This is actually true in arbitrary d dimensions, for modifications of the formulae see (Ryu and Takayanagi, 2006a,b).

10.3 A Massive Ising Screen?

We have now found a 3-dimensional gravity theory that is the holographic dual of the critical Ising model. But can we find a geometry describing the non-critical case as well? We present a couple of candidates.

10.3.1 Simple cutoff

In (Ryu and Takayanagi, 2006a), a calculation of the entanglement entropy for a massive theory is derived. This is done using the assumption that the mass introduces a correlation length ξ. A rough approximation of this feature is simply to say that no interactions are present at distances greater than ξ. If $l >> \xi$ this gives a cutoff in the integral for the geodesic distance and the result is:

$$S_A = \mathcal{A}\frac{c}{6}\log\frac{\xi}{a} \tag{10.9}$$

where $\mathcal{A}$ is the number of boundary points separating A from its complement B.

This result is a good approximation in the large mass limit $l >> \xi$, however, it is not exact, and should have correction terms due to the finite length l of A.[6]

10.3.2 The BTZ black hole

Apart from the empty AdS-space, the 3-dimensional Einstein equations with negative cosmological constant also admits black hole solutions. These solutions were first described by Bañados, Teitelboim and Zanelli in (Bañados et al., 1992). The geometry of this hole was described in (Bañados et al., 1993) and its geodesic structure in (Cruz et al., 1994). A more recent review of this is (Birmingham et al., 2001). Could the BTZ black hole be the place to look for the non-critical description?

In (Ryu and Takayanagi, 2006a), they calculate entanglement entropies for a non-spinning BTZ black hole. It is also shown that this is the entropy for a cylindrical dual theory. As argued in chapter 7, the cylinder case can also be viewed as a finite temperature case, and the temperature is in fact that of the BTZ black hole. However, this should not be confused with the non-critical theory, or as phrased by (Saluer and Bauer, 1989), the correspondence between BCFT and the non-critical theory *"…should not be confounded with the rather trivial observation that a critical model on a say periodic strip of width L due to the form of its transfer matrix …, can also be considered as a quantum one dimensional model of … inverse temperature* $\beta = 2\pi/L$*"*.

10.3.3 Topological massive gravity

Since the non-critical case is a pertubation of the critical one (sufficiently close to the critical point), by continuity we expect the dual geometry to be a pertubation of the original AdS-space. But which one?

One such candididate could be a topological massive gravity as described in for instance (Anninos et al., 2008). This is a theory with one Virasoro algebra, which is promising. However, the paper (Compère and Detournay, 2008) proves that such a theory has negative central charge, and hence corresponds to a non-unitary theory.

[6]The first few of these correction terms were calculated analytically in (Cardy et al., 2008). However, that calculation merely found the entanglement entropy from its definition and did not explore holographic duals.

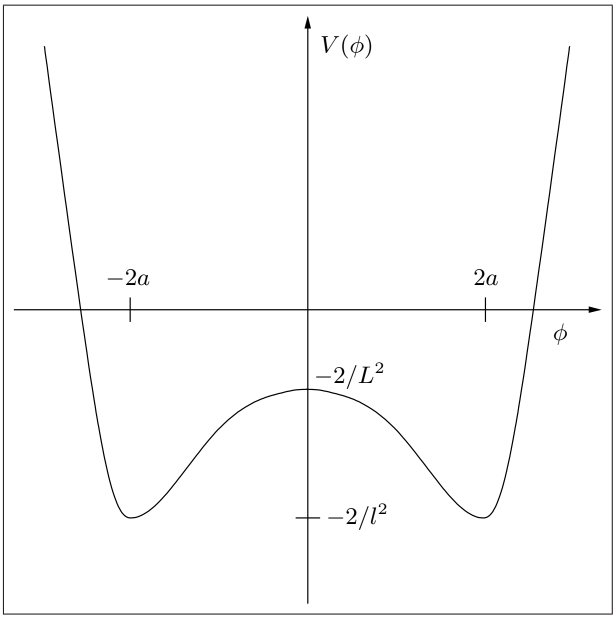

Figure 10.2: Schematic drawing of the shape of the potential $V(\phi)$ as a function of ϕ. Note that this potential is not the exact one used in (Hotta et al., 2008), rather it is a sketch showing the approximate shape and the placing of minima and local maximum points.

10.3.4 The CFT-interpolating black hole

The idea of a perturbation, however, might still bring the solution to our problem. In (Hotta et al., 2008), the authors use a perturbation to construct an interpolating solution between two CFTs in approximately the following manner.

(Hotta et al., 2008) couple the gravity to a scalar field ϕ with a potential $V(\phi)$. They then make a black-hole-like ansatz for the solution and choose a potential energy with local maximum at zero and local minima at $\phi = \pm 2a$ where a is a parameter.[7] This solution interpolates between the solution $\phi = 0$ and $\phi = 2a$.

Setting $\phi(\infty) = 0$ and $\phi(r_0) = 2a$, where r_0 is the Schwarzschild radius of the black hole, they obtain a CFT at the infinite boundary with

$$c_{UV} = \frac{3L}{2G_N}, \qquad L = \frac{a^2 l}{1 - e^{a^2}}, \tag{10.10}$$

where l is the curvature of the space at the Schwarzschild radius and L the curvature at infinity.

[7] A schematic drawing of the shape of the potential is shown in figure 10.2.

From this one also gets that

$$c_{IR} = \frac{3l}{2G_N} \tag{10.11}$$

The radius at which we choose our boundary holographic screen is an RG-parameter, so as we move from infinity into the black hole, we flow in RG-space from a CFT of central charge c_{UV} to central charge c_{IR}. In principle, one should then be able to calculate the geometrical entropy of a non-critical theory in the flow between the two by looking at geodesics between points at constant time at a radius corresponding to the theory's distance from criticality.

However, there is a problem with using this geometry to describe the non-critical Ising model. Looking at the two central charges c_{UV} and c_{IR}, we see that none can take the value 0 if they do not both take this value. The non-critical Ising model is a theory on a flowline from $c = 1/2$ to the trivial $c = 0$ theory. Just such a theory that cannot be described by the construction of (Hotta et al., 2008).

Nevertheless, the construction of a geometry interpolating between different conformal field theories is promising, and it is possible that a different potential, or possibly a different sort of perturbation[8] can give a holographic description of the Ising model.

In addition, the construction described in (Hotta et al., 2008) claims to give a holographic description of some $c_{IR} \neq 0$ theory. This is definitely interesting in its own right.

10.3.5 AdS → BCFT → non-critical theory

Another promising direction might be to take advantage of the identification of the non-critical theory with a boundary conformal theory. In this way, we might find an entropy for the massive case. However, it might not be the original entanglement entropy.

From (Ryu and Takayanagi, 2006a), we learnt that the non-spinning BTZ could describe the theory on a cylinder. If we could avoid identifying the endpoints of A, but rather make the boundary conditions free, and then exchange the modular parameter q for the nome of the non-critical theory, we should get a candidate entropy for the entire massive theory on the plane. From this we might be able to work out an entropy.

Doing this calculation is beyond the scope of this work. We will, however, give a rough sketch of how this calculation may be performed.

Imagine a locally AdS-like geometry with a conformal torus boundary.[9]

[8] A massive fermion, for instance?

[9] From (Birmingham et al., 2001), we know that the BTZ black hole is one such geometry. However, there are others, and luckily all such geometries were classified in (Maloney and Witten, 2007).

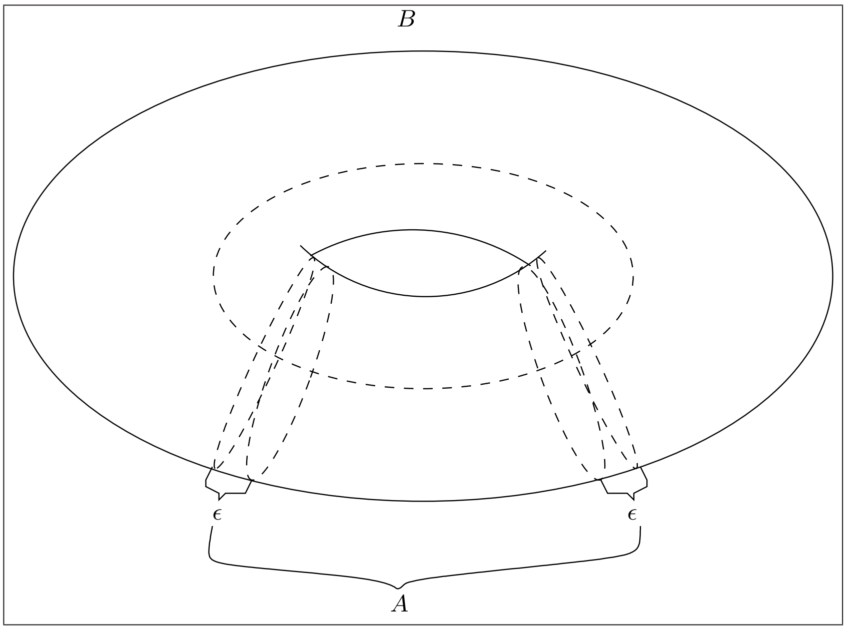

Figure 10.3: A schematic drawing of the BCFT construction on the torus-shaped conformal boundary of a locally AdS-like space. A is the subsystem that comprises the BCFT and it is the subsystem known to the observer. A in itself seems to be a boundary surface of one length l in the spatial direction, and a periodic length in the temporal direction determined by the geometry of the AdS-like space. Since the modular parameter is given by the ratio of these two lengths, we may tune it to our preferred value by tuning either l or the geometry of the dual space.

Proceed in the usual fashion to pick out a finite piece of the spatial dimension of length l and call this piece A and its complement B. Again choose a UV-cutoff smearing of the endpoints and denote it ϵ. Now, instead of identifying the endpoints of A by a periodic boundary condition, let the boundary conditions be free at both ends. Calculate the entanglement entropy. A schematic drawing of this geometric process can be seen in figure 10.3.

Now A has the appearance of a boundary annulus with free-free boundary conditions. The entanglement entropy we calculate then becomes a function of the modular parameter of A rather than of the geometry of the space-time and A's length. According to (Saluer and Bauer, 1989), by exchanging the modular parameter q of A with the nome q, we get a mapping to the integrable non-critical theory on the plane, and what started out as an entanglement entropy, is now some sort of entropy of non-criticality.

Fascinating as this may seem, there are several points at which this construction may fail. First, it might not be trivial to realise what imposing free rather than periodic boundary conditions will mean in terms of the geodesics of AdS. This may make the interpretation of the result in the dual theory difficult.

Secondly, though A looks at first sight to be a boundary surface, it might not be. Alhough B is unknown, it is still there, making it possible for the fields to continue into it. This is a severe threat indeed, and should be considered closely, since if A is not a boundary surface, this whole argument is nonsensical.

Finally, if the calculation of this quantity is meaningful and doable, an interpretation of this entanglement of non-criticality is needed. It may not be trivial, but it may also be a very trivial and well-known quantity already.

Chapter 11

Conclusion and Outlook

> 28. Denne boken er skrevet, med skrifttegn, med talltegn, med ord og setninger, med mellomrom mellom alle linjer, men også med tomrom både mellom alle ord og mellom alle tegnene,
>
> *Johannes' bok 33:28*, Georg Johannesen, 1978

In this thesis we have reviewed several calculational tools developed for 2-dimensional physics and applied them to the Ising model. We have tried to answer or partially answer the research questions listed in the introduction, and it is time to review our results.

11.1 Summary of Research Questions

1. What are the properties of effective field theory descriptions of critical 2-dimensional systems?

In chapter 4, we explained that a critical theory is also scale invariant. In two dimensions, this turned out to imply that the theory was invariant under conformal transformations. Hence, the effective field theory description of a critical 2-dimensional system must be given in terms of a CFT. In chapter 4, we gave a review of useful properties of such theories, and refer the reader who would like to delve deeper into this subject to such sources as (Di Francesco et al., 1997).

The results on CFTs are general and are invaluable tools in the study of all systems in two dimensions. The critical Ising model is no exception, and is described by a CFT of conformal charge $c = 1/2$

2. What restrictions are imposed on the flows in RG-space of 2-dimensional theories, and how can we calculate these flows?

In chapter 5 we stated and proved Zamolodchikov's C-theorem, which is in fact the answer to this question. We described its consequences for the RG-flow of theories on the 2-dimensional plane. This flow is in the direction of theories of lower values of the central charge. Hence, socalled limit cycles cannot occur in the RG-flow of theories on the 2-dimensional plane.

However, we saw that the proof cannot be extended to arbitrary geometry and dimension. We also looked into several attempts at generalisations and explanations why such generalisations are not sucsessful. It seems reasonable to conclude that Zamolodchikov's C-theorem is a feature of a special geometry rather than a general principle for all RG-flows.

The proof of Zamolodchikov's C-theorem gave us a way of calculating the C-function and we did so for the field theoretic version of the Ising model, Majorana fermions, in chapter 6. The resulting function exhibited the expected properties and was displayed in figure 6.1. Features of this calculation can in principle be used to calculate C-functions for general field theories on the 2-dimensional plane.

3. What is the effect of compactifying one or two of the dimensions?

In chapter 7 we looked at the effects of compactifying one dimension forming a cylinder. We calculated the vacuum energy for the Ising model on a cylinder both at and off the critical point, which provides a partial answer to this question.

We compactified the other direction as well in chapter 8, completing our answer. Again we calculated vacuum energies, but we also saw that the consistency on the torus poses constraints on the possible CFTs. In the Ising setting these constraints are not very important since only one choice of fields is possible. However, we saw that the ratios between contributions from different sectors was determined by modular invariance. This constraining of a theory by imposing modular invariance is an important CFT principle and is used for the general theory.

The vacuum energy of the non-critical Ising model computed in chapter 8 was shown to be consistent with that of the cylinder vacuum energy, as well as the conformal vacuum energy on the torus, and these quantities have not to the author's knowledge been calculated elsewhere. The calculations of vacuum energy for the critical cases both on the cylinder and torus were fairly straightforward, and are easily generalised to the general CFT case. Though the calculations of the non-critical vacuum energies were sometimes tedious, the methods involved should be applicable to other cases, although the theory in question should not be too complicated.[1]

4. What happens to a critical theory in 2-dimensions if the 2-dimensional geometry has boundaries? What sensible boundary conditions can we impose and how do they restrict the field theory content?

and

5. Which infinite symmetries are found away from criticality and are they somehow connected to the symmetry of the critical theories from which they flow?

The short answer to the first of these questions turned out to be that the theory is a BCFT with conformal boundary conditions, which we reviewed in sections 4.3.5 and 9.1. Simply put; the presence of a boundary constrains one Virasoro algebra to be some sort of mirror image of the other, breaking the double Virasoro algebra down to a single Virasoro algebra.

To the second question, the answer is integrability, which turns out to be a symmetry of one Virasoro algebra as well.

The main focus of chapter 9 was a very interesting relationship between these non-critical integrable theories and BCFTs for their corresponding critical theories. They turned out to have almost identical partition functions. The difference lay in the parameter q on which the partition function depended. In the BCFT case, q was a modular parameter of the annulus geometry upon which the theory resided. In the non-critical, integrable case, q could not be a geometrical parameter, since the theory was given on the full 2-dimensional plane. Instead q was given in terms of the theory's distance from criticality. We made calculations exemplifying this in the Ising setting.

We also looked at papers that tried to explain this relationship in a general manner. This review culminated in what appears to be a deep relationship between integrability and conformal symmetry.

On one side (Itoyama and Thacker, 1989) argued that integrability lead to a conformal symmetry constrained in the same way as a BCFT so that it contained only one Virasoro algebra. This constraint lifted in a natural way to the full conformal symmetry at the critical point.

On the other hand (Saluer and Bauer, 1989) proposed that one would always be able to perturb the critical theory so that it moved in an integrable direction, so that for each conformal theory there is an integrable non-critical theory with a partition function equal to that of its BCFT with appropriate boundary conditions.

[1] For instance, the partition function should be known or calculable.

It will be interesting to see whether this is indeed so, and whether the emerging integrable models correspond to realisable theories. If it is, it provides a very powerful tool to examine all integrable models, and indeed predict new ones. The incredible power of CFT, and by extension BCFT, is then at our disposal for a very broad range of theories.[2]

6. Can 2-dimensional theories capture the dynamics of the 3-dimensional spacetimes more adequately than Magritte could capture the properties of a 3-dimensional pipe on a 2-dimensional canvas?

Chapter 10 explored holography and the possibility of a mapping between a 2-dimensional theory and a 3-dimensional theory of gravity. We saw how this could be realised for the case of the critical Ising model and its mapping to a certain AdS-space. This particular calculation was made from a general formula, and can be performed just as easily for any CFT.

We also looked into attempts to construct geometries that could be the holographic duals of non-critical theories. A simple cutoff method was sufficient to yield first order results for a massive theory, but the insight gained from it was limited. The use of a BTZ black hole appeared to be too restricted, since it merely changed the geometry of the edge theory. A construction in terms of topological massive gravity was also found to be inapplicable, since it gave non-unitary dual theories.

The interpolating construction in (Hotta et al., 2008) could not be used for the non-critical Ising model. It should, however, be useful in other cases, and does point to a possibility for the Ising model as well. In the future, it will be interesting to see if an interpolating geometry corresponding to the non-critical Ising model, or indeed any non-critical theory flowing to the trivial $c = 0$ CFT, can be found.

We also discussed the posibility of using the BCFT/$(m \neq 0)$ correspondence discussed in chapter 9 to calculate some sort of non-critical entropy. Whether the quantity is calculable, or even meaningful, remains to be seen.

11.2 General Overview and Outlook

In this thesis, we have reviewed a lot of tools and new conceptual frameworks applicable to 2-dimensional physical systems. We have also applied these tools to the Ising model. Although it is the simplest 2-dimensional model, it is highly non-trivial, and calculability for the Ising model suggests that this type of analysis can be extended to more complicated systems.

[2]Provided we can indeed find the theory's distance from criticality, and from that infer the value of q as simply as proposed in (Saluer and Bauer, 1989).

Looking out on the landscape of 2-dimensional systems, we see that many things can be sorted out and classified. For instance, the RG flow does not form limit cycles, but flows steadily towards lower C-function values. It stops only at the critical points, where the theories can be described by CFTs, a feature which is very useful in the 2-dimensional case, since so much is known and calculable in 2-dimensional CFTs. For instance, vacuum energies on the cylinder and torus are obtainable.

To further delve into 2-dimensional theories on compact geometries, we have seen that key geometries are that of the torus and annulus. The constraints of modular invariance given by the torus consistency requirements are useful, and even more so for more complicated theories than for the Ising model. In a similar fashion, the Cardy conditions on the annulus leads to constraints on the content of a BCFT.

The simple relation between BCFTs and non-critical, integrable models allows us to turn a perturbative question into a question of geometry of a BCFT. This geometric viewpoint equips us with the power of CFT in the analysis of non-critical systems.

All this makes the landscape of 2-dimensional systems an intriguingly ordered one. Through the use of holography some of this order may be mapped into a 3-dimensional AdS world.

However beautiful this may look, many open questions remain.

First, although a calculation in the Ising model may point out a possible road to calculations on more complicated systems, such a calculation may not be possible in practice.

Secondly, the vacuum energy that we found for the critical Ising model in chapter 8, had the appearance of a combination of 2-forms of submodular groups. That this observation is in fact true, is not proved here, but should be proven. If it is, the question that remains is whether this is a general feature of torus CFT vacuum energies, and hence hints at a deep relationship between submodular groups and minimal conformal field theories.

Furthermore, although the correspondance between pure AdS-spaces and CFTs are straightforward, are there 3-dimensional analogues of all non-critical theories as well, and what are they? We have reviewed several possible constructs, whereof some are quite promising. The construction in (Hotta et al., 2008) seems especially useful, though it cannot describe the non-critical Ising model. A similar interpolating geometry that adequately describes theories flowing towards a trivial fixed point would be a natural direction for future research.

Finally, can the BCFT/integrable model correspondance applied to the holographic calculations be used to define an entropy in non-critical models? Is this quantity calculable, and does it make physical sense? It has not been within the scope of this thesis to answer these questions. In the future, however, the answers to them may turn out to be interesting and useful, or possibly quite the opposite.

Appendices

Appendix I

The Poisson Resummation Formula

Calculations of physical quantities on compact or partially compact geometries often lead to infinte sums. Manipulation and change of variables in such sums will then be very useful. In this appendix we state and prove the Poisson resummation formula which does this, and which we will be using in several calculations in this work.

The formula such as it is given here and the following proof may be found in (Green et al., 1987):

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 A + 2n\pi A x} = \frac{1}{\sqrt{A}} e^{\pi A x^2} \sum_{m=-\infty}^{\infty} e^{-\pi A^{-1} m^2 - 2i\pi m x} \tag{I.1}$$

We now use a property of the δ function, $\sum_m e^{2\pi i r m} = \sum_n \delta(r-n)$, to rewrite the left hand side of (I.1):

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 A + 2n\pi A x} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dr e^{2\pi i r m} e^{-\pi r^2 A + 2r\pi A x} \tag{I.2}$$

Performing the Gaussian integral of the right hand side of (I.2)[1] now proves its equality to the right hand side of (I.1).

[1]See for instance (Rottmann, 1995).

Appendix II

The Identification $\mathbb{R}^2 \to \mathbb{C}$

In this section we identify $\mathbb{R}^2$ with $\mathbb{C}$, and use this identification to get complexcoordinates z and $\overline{z}$. We also explain why this is a useful identification in 2-dimensional conformal field theories. The argument given is very close to the one given in section 1.2 of (Ginsparg, 1990).

Working in the 2-dimensional (x, y)-plane $\mathbb{R}^2$ we can always define the quantities

$$z = x + iy \qquad \overline{z} = x - iy$$

Each of these identifications give us a bijective mapping to $\mathbb{C}$. But z and $\overline{z}$ are not independent. However, we can let x and y be complex coordinates for a brief moment. We have then got a $\mathbb{C}^2$ plane, and the transition to z and $\overline{z}$ is just a change of basis. Keeping this point of view, we can postpone the identification $z^* = \overline{z}$ to the end of calculations . This condition assures that x and y are real, but it is also an equation restricting us to a complex line in $\mathbb{C}^2$. This line constitutes a $\mathbb{C}$ space, so $\mathbb{R}^2 \cong \mathbb{C}$.

We will now use the transition to the complex coordinates z and $\overline{z}$ to form a new basis, where we can calculate many quantities more readily. It is assumed that the reality condition on x and y, $z^* = \overline{z}$ must be imposed eventually.

In changing to the coordinates $z, \overline{z}$, we have to find the new metric. We know that:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{II.1}$$

Plugging the (x, y)-metric into equation (II.1) we find that

$$ds^2 = dx^2 + dy^2 \tag{II.2}$$

so the (x, y)-metric is $\delta_{\mu\nu}$.

For the $(z, \overline{z})$-metric we find

$$ds^2 = dz d\overline{z} \tag{II.3}$$

So

$$g_{zz} = g_{\overline{zz}} = 0 \qquad g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2} \tag{II.4}$$

Demanding that $g^{\mu\nu}$ be the inverse of $g_{\mu\nu}$ what we find is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Having found this more complicated metric, one might wonder why one would choose to use these complex coordinates. One good reason in a CFT-setting is the following. A conformal transformation $x^\mu \to x^\mu + \epsilon^\mu$ satisfies the following equation:[1]

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{II.5}$$

In the $(x, y) = (x^1, x^2)$ coordinates we get:[2]

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$$

We recognise these as the Cauchy-Riemann equations,[3] so that every conformal transformation is a holomorphic one:

$$z \to f(z) \qquad \overline{z} \to \overline{f}(\overline{z})$$

The change of metric is:

$$ds^2 = dz d\overline{z} \to \left| \frac{\partial f}{\partial z} \right|^2 dz d\overline{z}$$

Somtimes we will need the derivatives of x and y:

$$\frac{\partial}{\partial x} = \partial_z + \partial_{\overline{z}} \qquad \frac{\partial}{\partial y} = i(\partial_z - \partial_{\overline{z}}) \tag{II.6}$$

[1] See for instance section 1.1 of (Ginsparg, 1990).
[2] See section 1.2 of (Ginsparg, 1990).
[3] See section 2.1 of (Bak and Newman, 1997).

Appendix III

The Γ Function, ζ Functions and Bernoulli Polynomials

In this section we will list, and briefly study, some of the properties of a set of very important functions. These functions will occur in some of the models we will study, and it is therefore important for us to know something about their properties.

All of these functions are defined as partly diverging integrals or series diverging for certain values. We will look at the natural expansion of the functions to the complex plane for their area of convergence. From this we may expand the functions to unique meromorphic functions defined on all of $\mathbb{C}$ but for a finite, or at least countable, number of poles.

The properties we are interested in, are among others; formulas for the functions over the whole complex plane, location of the functions' singularities, zero-points and reflection formulas. Reflection formulas are formulas that give a relationship between the function value at the point z and a point $a - z$, thereby giving a value by reflection about the line $z = a$.[1]

III.1 The Γ Function

The Γ function is defined by the integral:[2]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{III.1}$$

[1]Normally this reflection formula is found because of the analytic continuation of the function. All of these functions are defined for $\Re z > a$ for some $a \in \mathbb{R}$. Analytic continuation is then used to extend the function to the rest of the complex plane. When a generalization of the Schwarz reflection principle is used, the reflection formula is what we get.

[2]The properties and definitions in this section can be found in (Bak and Newman, 1997).

From this, the Γ function is defined for the whole right half-plane, and we can also see that it has a singularity for z = 0.

Integration by parts gives us a recursion formula

$$\Gamma(z+1) = z\Gamma(z) \tag{III.2}$$

We also see, that $\Gamma(1) = 1$, and this combined with the recursion formula makes it evident that for $n \in \mathbb{N}, \Gamma(n) = (n-1)!$. This is a new motivation for our wanting to study this function. The Γ function is the expansion to the complex plane of the factorial.

We now use the recursion formula above to extend Γ to the complex half plane $\Re z > -k-1$. The formula is:

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)(z+2)...(z+k)} \quad \text{for} \quad \Re z > -k-1 \tag{III.3}$$

From this, we see that the singularities are isolated simple poles at the non-positive integers. The residues at these points are:

$$\mathrm{Res}(\Gamma(z); -k) = \frac{(-1)^k}{k!} \tag{III.4}$$

Using a slightly different approach to the extension process, involving a limiting process and integration by parts, we get an explicit product formula for $\Gamma(z)^{-1}$:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k} \tag{III.5}$$

where γ is the Euler constant.

$$\gamma = \lim_{n\to\infty} 1 + \frac{1}{2} + ... + \frac{1}{n} - \ln n \tag{III.6}$$

This formula leads us to the reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \tag{III.7}$$

We see from this that Γ has no zeroes and that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

III.1.1 The digamma function, ψ

The derivative of the Γ function is related to the digamma function ψ[3]:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{III.8}$$

[3]The properties of the digamma function (ψ) were found in (wikipedia, 2009a).

The digamma function has a simple recursion formula:

$$\psi(x+1) = \psi(x) + \frac{1}{x} \tag{III.9}$$

A special value worth noting is

$$\psi(1) = -\gamma \tag{III.10}$$

III.1.2 The β function

Another function closely related to the Γ function is the β function, defined as:[4]

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{for} \quad \Re(x), \Re(y) > 0 \tag{III.11}$$

We see that the β function is symmetric with respect to x and y. The connection to the Γ function is obvious in a slightly different form, namely:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{III.12}$$

From the properties of the Γ function, we see that the β function has isolated pole singularities for non-positive integer values of x or y. The function has a zero for x and y not non-positive integers in themselves, but the sum $x+y$ a non-positive integer.

III.2 The Bernoulli Polynomials, $B_n(x,y)$

Before we can start talking about the next of our exciting meromorphic functions, we need to know a thing or two about the Bernoulli polynomials.[5]

The Bernoulli polynomials are a familiy of polynomials, given by:

$$B_n = \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k} \tag{III.13}$$

for $n \leq 0$ and b_k the Bernoulli numbers. The Bernoulli numbers are generated by the zero-point values of the Bernoulli polynomials respectively. A few select values are $B_2 = 1/6$, $B_4 = -1/30$.

[4]The properties of the β function listed below, can be found in for instance (wikipedia, 2009b).

[5]The properties of the Bernoulli polynomials listed below may be found in, for instance, (wikipedia, 2009c).

Summing up the Bernoulli polynomials times a factor $t^n/n!$ gives us the interesting formula:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{III.14}$$

Doing an inverse Fourier transform on this, we can get a formula for the Bernoulli polynomials expressed in terms of the differential operator $D = d/dx$:

$$B_n(x) = \frac{D}{e^D - 1}x^n \tag{III.15}$$

This gives us an explicit formula for calculating any Bernoulli polynomial. It is also known that the Bernoulli polynomial gives formulas for integer values of one of the functions we shall look at, namely the Hurwitz zeta function (ζ_H) with special restriction to the Riemann zeta function (ζ_R):

$$B_n(x) = -n\zeta_H(1 - n, x) \tag{III.16}$$

III.3 ζ Functions

A general ζ function is defined by an infinite series of the form (valid where this series converges):

$$\zeta(z, a) = \sum_{n=1}^{\infty} \alpha(n, a)^z \tag{III.17}$$

α is a function that fully specifies the member of the ζ function family. The series definition can be extended analytically to yield a meromorphic function over the whole complex plane for any sensible α.

There is obviously an infinitly large family of such functions. We will only take a closer look at a few of the most famous family members:

III.3.1 The Riemann ζ function (ζ_R)

The Riemann ζ function is probably the most extensively studied, and simplest of the ζ functions. In many texts it is simply called the ζ function, and it was the first ζ function to be studied. It is given simply by $\alpha(n) = \frac{1}{n}$, i.e.[6]

[6] The properties of the Riemann ζ function and the prime number theorem were found in (Bak and Newman, 1997).

$$\zeta_R(z) = \sum_{n=1}^{\infty} (n)^{-z} \tag{III.18}$$

We see that the series expansion for ζ_R converges for $\Re z > 1$. One of the most interesting and baffeling properties of ζ_R is that it can be fully defined by all the primes. Although this property is amazing, it is not difficult to realise from the definition of it. Observing that:

$$\frac{1}{2^z}\zeta_R(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + ...$$

So

$$(1 - \frac{1}{2^z})\zeta_R(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + ...$$

In a similar way we can use the unique prime factorization of the integers to take out all the parts of ζ_R given by $1/(ap)^z$ for each prime number p. Hence we get:

$$\prod_{p \text{prime}} \left(1 - \frac{1}{p^z}\right) \zeta_R(z) = 1$$

Or equivalently:

$$\zeta_R = \frac{1}{\prod_{p \text{prime}} \left(1 - \frac{1}{p^z}\right)} \tag{III.19}$$

From this we see that ζ_R has a singularity at $z = -1$. When extending ζ_R beyond $\Re z > 1$, one can see that this is the only singularity of ζ_R in the whole complex plane. An explicit such formula is:

$$\zeta_R = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} \tag{III.20}$$

It can also be proven that ζ_R is zero-free for $\Re z \leq 1$. From this property, and the close relationship between ζ_R and the primes we can prove the famous prime-number theorem:

The Prime Number Theorem III.3.1 *Let $\pi(N)$ be the number of primes less than or equal to N. Then $\pi(N)$ is proportional to $N/\ln N$ that is:*

$$\frac{\pi(N)\ln N}{N} \to 1 \text{ as } N \to \infty$$

There is a reflection formula for ζ_R involving the Γ function. This formula also reveals a close relationship between the Γ and ζ_R functions:

$$\zeta_R(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta_R(1-z) \tag{III.21}$$

From this function, we can easily see that ζ_R has zeros for all negative even integers. These zeros are called the trivial zeros of the Riemann ζ function. However, these are not the only zeros of ζ_R, and the other zeros are all the more interesting. It can be shown that all of the non-trivial zeros lie within the so-called critical strip: $z \in \mathbb{C} : 0 < \Re z < 1$. The famous, but yet unproven, Riemann hypothesis states that all of these zeros are to be found on the critical line $z \in \mathbb{C} : \Re z = \frac{1}{2}$.

III.3.2 The Hurwitz ζ function (ζ_H)

The Hurwitz ζ function, can be seen as a generalization of the Riemann ζ function. ζ_H is a function of two variables a and z, and the function α is given $\alpha(n,a) = (n+a)^{-1}$, so there is effectively a shift from integers to some general numbers with integer separations in the summation. I.e. ζ_H is given by:[7]

$$\zeta_H(z,a) = \sum_{n=0}^{\infty} (n+a)^{-z} \tag{III.22}$$

For rational values of a, $a = m/n$ s.t. $m, n \in \mathbb{Z}, n \neq 0$, the reflection formula is

$$\zeta_H\left(1-z, \frac{m}{n}\right) = \frac{2\Gamma(z)}{(2\pi n)^z} \sum_{k=1}^{n} \cos\left(\frac{\pi z}{2} - \frac{2\pi k m}{n}\right) \zeta_H\left(z, \frac{k}{n}\right) \tag{III.23}$$

We also have an interesting relationship between ζ_H for negative integer values of z, and the Bernoulli polynomials $B_n(x)$:

$$\zeta_H(-n,x) = -\frac{B_{n+1}}{n+1} \tag{III.24}$$

The expression

$$\zeta_H(0,a) = -a + \frac{1}{2} \tag{III.25}$$

will also be useful.

III.3.3 Derivatives and the ζ-regularisation scheme

As well behaved ζ functions may be analytically continued to some part of $\mathbb{C}$, it is natural to be interested in their derivatives. Within the radius of

[7] The properties of ζ_H listed here may be found in (wikipedia, 2009d).

convergence of the formal sum, we may simply take the derivative of each term:

$$\frac{\partial}{\partial z}\zeta(z,a) = \sum_{n=1}^{\infty} \alpha(n,a)^z \ln \alpha(n,a) \tag{III.26}$$

Then an analytic continuation may be performed as usual though the radius of convergence may be smaller.

This expression for the derivative of the ζ functions gives us a way to regularise formally infinte sums. Consider for instance:

$$\sum_{n=1}^{\infty} f(n) \tag{III.27}$$

Define

$$\zeta_f(z) = \sum_{n=1}^{\infty} \left(e^{f(n)}\right)^s \tag{III.28}$$

then

$$\zeta_f'(0) = \sum_{n=1}^{\infty} f(n) \tag{III.29}$$

Specifically[8]

$$\zeta_R'(0) = -\sum_{n=1}^{\infty} \ln(n) = -\frac{1}{2}\ln(2\pi) \tag{III.30}$$

and[9]

$$\zeta_H'(0,a) = \ln\Gamma(a) - \frac{1}{2}\ln 2\pi \tag{III.31}$$

In this thesis we look at physics on compact or partially compact geometries. In such cases, what would otherwise be functional integrals are turned into infinite sums. Here the ζ function regularisation will be especially useful.

[8]This derivative, and many other ζ_R properties, may be found in (mathworld, 2009a).
[9]This derivative, and many other ζ_H properties, may be found in (mathworld, 2009b).

Appendix IV

Bessel Functions

In this text we will make use of the modified Bessel functions of the second kind. This section will show briefly how they arise in calculations, and list some of their most important properties.[1]

IV.1 Generalities

The modified Bessel functions of the second kind are solutions of the modified Bessel differential equation together with the modified Bessel function of the first kind.

$$x^2 y'' + xy' - \left(x^2 + n^2\right) y = 0$$

Its relationship with the Hankel function is reminiscient of the relationship between trigonometric and hyperbolic functions.

$$K_n(x) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix) = \frac{\pi}{2} i^{n+1} \left(J_n\left(ix\right) + iY_n\left(ix\right)\right)$$

IV.2 Connection to the Majorana Propagator

In our case, the modified Bessel function of the second kind occurs in the propagator of the Majorana fermion, and knowing its properties will therefore be highly useful. Starting with the Majorana propagator, we will now see why this is so:

[1]Most of the properties were found in interactive sources such as (efunda, 2009). The derivation in section IV.2 is based mostly on quite trivial relations that may be found such places as (Rottmann, 1995), though one relationship was also found in (Gradshteyn and Ryzhik, 2000). This will be specified when the relationship in question is used. The definition of the Majorana propagator is to be found in most books on field theory, for instance in (Peskin and Schroeder, 1995). The $x \to 0$ behaviour of the K_0 and K_1 were found in (wikipedia, 2009e).

$$\Delta(x;m) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{-ip\cdot x}}{p^2+m^2} \tag{IV.1}$$

where d is the dimension of the space, m is the mass, and p is the momentum.

To perform this integral, let x be the length of the position vector. We can then write the propagator as:

$$\Delta(x;m) = \frac{S_{d-1}}{(2\pi)} \int_0^\infty dp p^{d-1} \int_0^\pi d\theta \sin^{d-2}\theta \frac{e^{-ipx\cos\theta}}{p^2+m^2} \tag{IV.2}$$

where

$$S_d = \frac{1}{(2\pi)^d} \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \tag{IV.3}$$

Define

$$\nu = \frac{d}{2} - 1 \qquad z = px$$

Let us look at the θ integration seperately for a moment. Defining:

$$I(z) = \int_0^\pi d\theta \sin^{2\nu}\theta e^{-iz\cos\theta} \tag{IV.4}$$

We may divide the integral (IV.4) into two parts, integrating first to $\pi/2$ then from there to π. Using the following properties of the trigonometric functions

$$\cos(\pi-\theta) = -\cos\theta \qquad \sin(\pi-\theta) = \sin\theta$$

enables us to change the boundaries of the last part of the integral, finally obtaining:

$$I(z) = \int_0^{\frac{\pi}{2}} d\theta \sin\theta^{2\nu} \left(e^{-iz\cos\theta} + e^{iz\cos\theta}\right) \tag{IV.5}$$

so

$$I(z) = 2\int_0^{\frac{\pi}{2}} d\theta \sin\theta^{2\nu} \cos(z\cos\theta) \tag{IV.6}$$

Now we can use the following definition of the Bessel function of the first kind:

$$J_\nu(x) = \frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(x\cos t)\sin^{2\nu} t dt \tag{IV.7}$$

so

$$I(z) = \sqrt{\pi} J_\nu(z) \left(\frac{2}{z}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \tag{IV.8}$$

Plugging all the above results into the propagator (IV.1), yields:

$$\Delta(x;m) = \frac{1}{(2\pi)^{\nu+1}} x^{-\nu} \int_0^\infty \frac{dp}{p^2+m^2} p^{\nu+1} J_\nu(px) \tag{IV.9}$$

Finally we use the following relationship between the modified Bessel functions of the second kind and the Bessel functions of the first kind:[2]

$$\int_0^\infty dx \frac{J_\nu(bx) x^{\nu+1}}{(x^2+a^2)^{\mu+1}} = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab) \tag{IV.10}$$

Applying equation (IV.10) to (IV.9), and inserting the value of ν, we see that the value of $\Delta(x;m)$ can be given in terms of Bessel functions K_ν as follows:

$$\Delta(x;m) = \frac{m^{d-2}}{(2\pi)^{\frac{d}{2}}} \frac{K_{\frac{d}{2}-1}(mx)}{(mx)^{\frac{d}{2}-1}} \tag{IV.11}$$

In the 2-dimensional case this becomes:

$$\Delta(x;m) = \frac{1}{2\pi} K_0(mx) \tag{IV.12}$$

IV.3 Derivatives and limiting behaviour

For practical purposes like calculating correlators, we need to know the derivatives of these functions and the realtions between different n-level functions.

In fact, the level n modified Bessel function of the second kind is given by a difference equation:

$$K_{n+1}(x) = K_{n-1}(x) + \frac{2n}{x} K_n(x) \tag{IV.13}$$

We can then express all the functions in terms of K_0 and K_1.

In particular:

$$K_2(x) = K_0(x) + \frac{2}{x} K_1(x)$$

There are many general expressions for the derivative. Here we will make use of the expressions

$$K_n'(x) = -K_{n-1}(x) - \frac{n}{x} K_n(x) \tag{IV.14}$$

or

[2]This relation may be found in equation 4 of section 6.565 of (Gradshteyn and Ryzhik, 2000).

$$K_n'(x) = \frac{n}{x}K_n(x) - K_{n+1}(x) \tag{IV.15}$$

Using (IV.14) and (IV.15), we find the derivatives of K_0 and K_1 which is all we actually need:

$$K_0'(x) = -K_1(x) \qquad K_1'(x) = -K_0(x) - \frac{1}{x}K_1(x) \tag{IV.16}$$

In addition we need to know the behaviour of these functions when $x \to 0$.

$$\lim_{x\to 0} K_0(x) = -\ln\left(\frac{x}{2}\right) \tag{IV.17}$$

$$\lim_{x\to 0} K_1(x) = \frac{1}{|x|} \tag{IV.18}$$

Appendix V

Automorphic Functions and Forms

In this thesis we will do a lot of work on the torus. As seen in section 3.2.3, scaling invariance on the torus naturally implies modular invariance or invariance under the discrete group $\mathrm{PSL}(2, \mathbb{Z})$ generated by translation ($\mathcal{T}$) and inversion ($\mathcal{S}$).

Below, we will list and explore certain properties of functions with special transformation properties under this group or some of its subgroups. These functions and forms and their relations to the torus are quite analogous to the trigonometric functions and their relations to the sircle.

V.1 Jacobi θ Functions and Dedekind's η Function

In this section we will give a description and some useful properties of the Jacobi θ functions. All these properties can be found in (Di Francesco et al., 1997) as well as in numerous tables of mathematical functions and their properties. As mentioned earlier they are useful for calculations on the torus because of their quasi-double-periodic properties.

Starting with the definitions, the four θ functions are defined as follows:

$$\theta_1(z|\tau) = -i \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-1/2} y^r q^{r^2/2} \tag{V.1}$$

$$\theta_2(z|\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} y^r q^{r^2/2} \tag{V.2}$$

$$\theta_3(z|\tau) = \sum_{n\in\mathbb{Z}} y^n q^{n^2/2} \tag{V.3}$$

$$\theta_4(z|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n y^n q^{n^2/2} \tag{V.4}$$

where $q = \exp 2\pi i\tau$, $y = \exp 2\pi i z$. The Jacobi triple product identity:

$$\prod_{r\in\mathbb{N}+\frac{1}{2}}^{\infty}(1+q^r t)(1+q^r/t) = \prod_{n=1}^{\infty}\frac{1}{1-q^n}\sum_{n\in\mathbb{Z}} q^{n^2/2}t^n \tag{V.5}$$

allow us to rewrite equations (V.1), (V.2), (V.3) and (V.4):

$$\theta_1(z|\tau) = -iy^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}\left(1-yq^{n+1}\right)\left(1-y^{-1}q^n\right) \tag{V.6}$$

$$\theta_2(z|\tau) = y^{1/2}q^{1/8}\prod_{n=1}^{\infty}(1-q^n)\prod_{n=0}^{\infty}\left(1+yq^{n+1}\right)\left(1+y^{-1}q^n\right) \tag{V.7}$$

$$\theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+\frac{1}{2}}(1+yq^r)\left(1+y^{-1}q^r\right) \tag{V.8}$$

$$\theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)\prod_{r\in\mathbb{N}+\frac{1}{2}}(1-yq^r)\left(1-y^{-1}q^r\right) \tag{V.9}$$

Before giving the modular transformation properties of the θ functions, we also give the definition of Dedekind's η function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{V.10}$$

Defining $\theta_n(\tau) = \theta_n(0|\tau)$, the transformation properties of θ functions 2, 3 and 4, and Dedekind's η function under $\mathcal{T} : \tau \to \tau + 1$ and $\mathcal{S} : \tau \to -1/\tau$ are:

$$\begin{aligned}
\theta_2(\tau+1) &= e^{i\pi/4}\theta_2(\tau) && \text{(V.11)}\\
\theta_3(\tau+1) &= \theta_4(\tau) && \text{(V.12)}\\
\theta_4(\tau+1) &= \theta_3(\tau) && \text{(V.13)}\\
\eta(\tau+1) &= e^{i\pi/12}\eta(\tau) && \text{(V.14)}\\
\theta_2(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_4(\tau) && \text{(V.15)}\\
\theta_3(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_3(\tau) && \text{(V.16)}\\
\theta_2(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_2(\tau) && \text{(V.17)}\\
\eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\eta(\tau) && \text{(V.18)}
\end{aligned}$$

In the limit $(z|\tau) \to (0|i\infty)$ we see from (V.6), (V.7), (V.8) and (V.9) that:

$$\lim_{\Im\tau\to\infty} \theta_1(0|\tau) = 0 \quad \theta_2(0|\tau) = 0 \quad \theta_3(0|\tau) = 1 \quad \theta_4(0|\tau) = 1 \tag{V.19}$$

V.2 Jacobi Elliptic Functions

The θ functions can be used to define the Jacobi elliptic functions. We also list some simple properties.[1]

Define first $\theta_n = \theta_n\left(0|\sqrt{q}\right) \quad n = 1, 2, 3, 4$, q is the nome of the elliptic functions to be defined in (V.27) and $k = (\theta_2/\theta_3)^2$ and $u = \pi\theta_3^2 z$:

$$\mathrm{sn}(u|k) = \frac{\theta_3\theta_1(z|\sqrt{q})}{\theta_2\theta_4(z|\sqrt{q})} \tag{V.20}$$

$$\mathrm{cn}(u|k) = \frac{\theta_4\theta_2(z|\sqrt{q})}{\theta_2\theta_4(z|\sqrt{q})} \tag{V.21}$$

$$\mathrm{dn}(u|k) = \frac{\theta_4\theta_3(z|\sqrt{q})}{\theta_3\theta_4(z|\sqrt{q})} \tag{V.22}$$

Actually the elliptic functions may be seen as a sort of inverses of the incomplete elliptic integral of the first kind. If u is set equal to the incomplete elliptic integral of the first kind:

$$u = F(\phi, k) = \int_0^\phi \frac{dt}{\sqrt{1-k^2\sin^2 t}} \tag{V.23}$$

then

$$\mathrm{sn}(u|k) = \sin\phi \tag{V.24}$$

$$\mathrm{cn}(u|k) = \cos\phi \tag{V.25}$$

$$\mathrm{dn}(u|k) = \sqrt{1-k^2\sin^2\phi} \tag{V.26}$$

The complete elliptic integral $K(k) = F(\pi/2, k)$ is also called the real quarter period whereas $K' = K(\sqrt{1-k^2})$ is called the imaginary quarter period. Together they define the nome of the elliptic functions,

$$q = e^{-\frac{\pi K'}{K}} \tag{V.27}$$

which is often used to express the elliptic functions in convenient forms.

[1]The properties listed here can be found in (mathworld, 2009c) and (mathworld, 2009d).

V.3 Eisensteine – Automorphic Forms

The Jacobi θ functions and the Dedekind η function are not the only objects with special transformation properties under the $\mathrm{PSL}(2,\mathbb{Z})$. We can actually form infinite chains of objects with special transformation properties.

These can be viewed as k-tensors or *modular forms* of weight $w = 2k$ of the modular holomorphic geometry.[2] The modular forms we will focus on here are the Eisensteine:

$$E_{2k}(z) \equiv N_k \sum_{\mathbb{Z}\times\mathbb{Z}}{}' (nz+m)^{-2k} \tag{V.28}$$

where N_k is a normalization and the sum is over the 2-dimensional lattice of integers (m,n) except for the point at the origin.

This can be rewritten in terms of the $q = \exp(2\pi i z)$ as

$$E_{2k}(z) = 1 - \frac{4k}{b_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n} \tag{V.29}$$

where b_{2k} are the Bernoulli numbers.[3]

When the z is transformed with a modular transformation $\gamma : z \to (az+b)/(cz+d)$ the E_{2k} for $k > 1$ transform according to:

$$E_{2k} \to E_{2k}(z') = (cz+d)^{2k} E_{2k}(z) \tag{V.30}$$

and this can be seen from a resummation of the transformed function. This means that the E_{2k} for $k > 1$ are $2k$-forms.

However, for $k = 1$ the sum in (V.29) does not converge uniformely and this resummation cannot be done. The transformation property for this Eisenstein function is:

$$E_2 \to E_2(z') = (cz+d)^2 E_2(z) - \frac{6ic}{\pi}(cz+d) \tag{V.31}$$

making it proportional to a connection rather than a tensor.

V.3.1 Eisenstein series at higher level

The modular group is rather a large group, and it will prove interesting to look at subgroups of it. We can for instance form the subgroup $\Gamma(N)$ at level N in the following way:

[2]See (Lütken, 2006) for a review of the automorphic forms and how they arise naturally in the context of modular invariance and hyperbolic geometry. Most of the facts listed in this section can be found there. Another useful source for this material is (Schoeneberg, 1974).

[3]See section III.2.

$$\Gamma(N) = \gamma \in \Gamma(1) \quad \text{s.t.} \quad \gamma = I \mod N \tag{V.32}$$

where $\Gamma(1)$ is the entire modular group, $\mathrm{PSL}(2,\mathbb{Z})$.

We also introduce the terminology that $\Gamma \subset \Gamma(1)$ is a subgroup at level N if N is the smallest number s. t. $\Gamma(N) \subset \Gamma$.

We immediately see that $\Gamma(1)$ is the only subgroup at level 1. However, we shall mainly be interested in the subgroups at level 2.

At level 2 there are in fact four subgroups: $\Gamma(2)$, and the subgroups generated by $\Gamma(2)$ and $\mathcal{T}$, $\mathcal{S}$ or $\mathcal{W}$ respectively. Where $\mathcal{T} : z \to z+1$, $\mathcal{S} : z \to -1/z$ and $\mathcal{W} = \mathcal{TST}$. The latter three are named for their unique generators, so they are $\Gamma_T = \Gamma_0(2)$, $\Gamma_S = \Gamma_\theta(2)$ and $\Gamma_W = \Gamma^0(2)$ respectively.[4]

We can form the Eisensteine belonging to these groups by taking certain linear combinations of parts of the original Eisensteine, in the following way:

$$E_{2k}^{(a,b)}(z) \equiv N_k \sum_{\substack{m=a \mod 2 \\ n=b \mod 2}}{}' (mz+n)^{-2k} \tag{V.33}$$

We will now only look at the weight $2k = 2$ Eisensteine. These functions transform like connections under different subgroups. $E_2^{(0,1)}$ is proportional to a connection under Γ_T, $E_2^{(1,1)}$ is proportional to a connection under Γ_S and $E_2^{(1,0)}$ is proportional to a connection under Γ_W. $E_2^{(0,0)}$ is proportional to a connection under all of them, since it is under $\Gamma(1)$.[5]

In (Schoeneberg, 1974) it is proven that linear combination

$$E^{(a,b)}(z) - E^{(0,0)}(z) \tag{V.34}$$

is a tensor of weight two under Γ_R if Γ_R is the subgroup of level 2 under which $E^{(a,b)}(z)$ is proportional to a connection. We will choose to define E^R to be 2 times this linear combination.[6]

After a few lines of calculations, one realises that

$$E_2^T(z) = 2E(2z) - E(z) = 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1+q^n} \tag{V.35}$$

Finding similiar expressions for E_2^S and E_2^W turns out to be more complicated, and will not be needed here.

---

[4]The second names of these and their origins can be found in (Schoeneberg, 1974).

[5]This can easily be seen from the fact that $E_2(z)$ has this property. Taking out a common factor of 2 from both n and m gives a summation over the entire lattice and we realise that $E_2^{(0,0)} = E_2/4$.

[6]Every linear combination of tensors are tensors.

Index

Bibliography

Ian Affleck. Universal term in the free energy at a critical point and the conformal anomaly. *Phys. Rev Lett. 56*, pages 746–748, 1986.

Dionysios Anninos, Wei Li, Megha Padi, Wei Song, and Andrew Strominger. Warped ads_3 black holes. *arXiv:hep-th/08073040v1*, 2008.

Joseph Bak and Donald J. Newman. *Complex Analysis Second Edition*. Springer-Verlag New York, Inc, 1997. ISBN 0-387-94756-6.

Rodney J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press Inc., London, 1982. ISBN 0-12-083180-5.

Máximo Bañados, Claudio Teitelboim, and Jorge Zanelli. Black hole in three-dimensional spacetime. *Phys. Rev Lett. 69*, pages 1849–1851, 1992.

Máximo Bañados, Marc Henneaux, Claudio Teitelboim, and Jorge Zanelli. Geometry of the 2 +1 black hole. *Phys. Rev D 48*, pages 1506–1525, 1993.

Alexander A. Belavin, Alexander M. Polyakov, and Alexander B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B241*, page 333, 1984.

Danny Birmingham, Ivo Sachs, and Siddhartha Sen. Exact results for the btz black hole. *arXiv:hep-th/0102155v2*, 2001.

H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale. Conformal invariance, the central charge, and universal finite-size amplitudes at criticality. *Phys. Rev. Lett. 56*, pages 742–745, 1986.

Pasquale Calabrese and John Cardy. Entanglement entropy and quantum field theory. *Journal of Statistical Mechanics: Theory and Experiment P06002*, 2004.

Andrea Cappelli, Daniel Friedan, and José I. Latorre. c-theorem and spectral representation. *Nucl. Phys. B352*, pages 616–670, 1991.

J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon. Form factors of branch-point twist fields in quantum integrable models and entanglement entropy. *J. Stat. Phys. 130*, pages 129–168, 2008.

John Cardy. Boundary conformal field theory. *arXiv:hep-th/0411189v2 20. Feb 2008*, 2008.

John L. Cardy. Conformal invariance and statistical mechanics. In *Les Houches Session XLIX 1988; Champs, cordes et phénomènes critiques; Fields, strings and critical phenomena*, pages 169–245. North-Holland, Elsevier Science Publishers B. V., 1990. ISBN 0-444-88440-8.

Geoffrey Compère and Stéphane Detournay. Semi-classical central charge in topologically massive gravity. *arXiv:hep-th/08081911v2*, 2008.

Norman Cruz, Cristián Martínez, and Leda Peña. Geodesic structure of the (2 + 1)-dimensional btz black hole. *Class. Quantum Grav. 11*, pages 2731–2739, 1994.

E. Date, M. Jimbo, T. Miwa, and M. Okado. Fusion of the eight vertex sos model. *Lett. Math. Phys 12*, pages 209–215, 1986.

E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado. Exactly solvable sos models: Local height probabilities and theta function identities. *Nucl. Phys. B290*, pages 231–273, 1987.

E. Date, M. Jimbo, T. Miwa, and M. Okado. Solvable lattice models. In *Proceedings of Symposia in Pure Math. 49*, pages 295–331. American Mathematical Society, 1989.

Brian Davies and Paul A. Pearce. Conformal invariance and critical spectrum of corner transfer matrices. *J. Phys A 23*, pages 1295–1312, 1990.

Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. Springer-Verlag New York, Inc, 1997. ISBN 0-387-94785-X.

efunda. Modified bessel function, 2009. URL `http://www.efunda.com/math/bessel/modifiedbessel.cfm`. The site was checked last on May 8th 2009.

D. Friedan, Z. Qui, and S. Shenker. Conformal invariance, unitarity, and critical exponents in two dimensions. *Phys. Rev. Lett 52*, pages 1575–1578, 1984.

Daniel Friedan. Lecture at the nordisk forskarkurs, recent developements in quantum field theory. 1990.

Daniel Friedan. Email correspondance to josé latorre. 1990/1991?

Stephen H. Friedberg, Arnold J. Insel, and Lawrence Spence. *Linear Algebra, Fourth Edition*. Pearson education, Inc., Upper Saddle River, New Jersey 07458, 2003. ISBN 0-13-120266-9.

José Gaite. Relative entropy in 2d quantum field theory, finite-size corrections and irreversibility of the renormalization group. *Phys. Rev. Lett. 81*, pages 3587–3590, 1998.

José Gaite. Entropic c theorems in free and interacting two-dimensional field theories. *Phys. Rev. D 61*, page 045006, 2000.

José Gaite and Denjoe O'Connor. Field theory entropy, the h theorem, and the renormalization group. *Phys. Rev. D 54*, pages 5163–5173, 1996.

Joachim Gasquet. *Joachim Gasquet's Cézanne, - a Memoir with Conversations*. Thames and Hudson, 1991. ISBN 9780500092125.

Paul Ginsparg. Applied conformal field theory. In *Les Houches Session XLIX 1988; Champs, cordes et phénomènes critiques; Fields, strings and critical phenomena*, pages 1–168. North-Holland, Elsevier Science Publishers B. V., 1990. ISBN 0-444-88440-8.

Stanislaw D. Glazek and Kenneth G. Wilson. Limit cycles in quantum theories. *arXiv:hep-th/0203.088v1*, 2002.

I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products, sixth edition*. Academic Press, 2000. ISBN 0-12-294757-6. Edited by Alan Jeffrey and Dan Zwillinger and translated from russian by Scripta Technica, Inc.

Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring theory; Volume 2; Loop amplitudes, anomalies and phenomenology*. Cambridge University Press, 1987. ISBN 0-521-35753-5.

Oeyvind Groen and Sigbjoern Hervik. *Einstein's General Theory of Relativity. With Modern Applications in Cosmology*. Springer Science + Buisness Media, LLC, 2007. ISBN 978-0-387-69199-2.

Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002. ISBN 0-521-79540-0. The book was also availiable for free on the authors home page http://www.math.cornell.edu/ hatcher/AT/ATpage.html at least up until the 2. of february 2009.

Per Holm. *Topologiske rom et cetera*. 2004. This version was availiable for free on the web from the adress http://www.uio.no/studier/emner/matnat/math/MAT4500/h06/pensumliste.xml at least up until the 2. of february 2009.

Christoph Holzhey, Finn Larsen, and Frank Wilcek. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys B424*, pages 443–467, 1994.

Kyosuke Hotta, Yoshifumi Hyakutake, Takahiro Kubota, Takahiro Nishinaka, and Hiroaki Tanida. The cft-interpolating black hole in three dimensions. *arXiv:hep-th/08110910v3*, 2008.

H. Itoyama and H. B. Thacker. Lattice virasoro algebra and corner transfer matrices in the baxter eight-vertex model. *Phys. Rev. Lett. 58*, pages 1395–1398, 1987.

H. Itoyama and H. B. Thacker. Integrability and virasoro symmetry of the noncritical baxter/ising model. *Nucl. Phys. B 320*, pages 541–590, 1989.

Claude Itzykson and Jean-Michel Drouffe. *Statistical field theory Volume 2*. Cambridge University Press, 1989. ISBN 0-521-37012-4.

Bjoern Jahren. *Differential geometry on surfaces*. 2007a. This version was availiable for free on the web from the adress http://folk.uio.no/bjoernj/kurs/4510/dg.pdf at least up until the 12. of november 2008.

Bjoern Jahren. *An introduction to hyperbolic geometry*. 2007b. This version was availiable for free on the web from the adress http://folk.uio.no/bjoernj/kurs/4510/hypeng.pdf at least up until the 2. of february 2009.

Henrik Johannesson. Lattice virasoro algebras, unitarity and the eight-vertex model. *J. Math. Phys 33*, pages 1–3, 1992.

Igor R. Klebanov and Juan M. Maldacena. Solving quantum field theories via curved spacetimes. *Physics Today, January 2009*, pages 28–33, 2009.

Carsten Andrew Lütken. Holomorphic anomaly in the quantum hall system. *Nucl. Phys. B 759*, pages 343–369, 2006.

Juan M. Maldacena. The large n limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys. 2*, pages 231–252, 1998.

Alexander Maloney and Edward Witten. Quantum gravity partition functions in three dimensions. *arXiv:hep-th/0712.0155v1*, 2007.

Franz Mandl and Graham Shaw. *Quantum Field Theory, Revised Edition*. John Wiley and Sons Ltd., 2007. ISBN 978-0471-94186-6.

mathworld. Riemann zeta function, 2009a. URL `http://mathworld.wolfram.com/RiemannZetaFunction.html`. The site was checked last on May 8th 2009.

mathworld. Hurwitz zeta function, 2009b. URL `http://mathworld.wolfram.com/HurwitzZetaFunction.html`. The site was checked last on May 8th 2009.

mathworld. Jacobi elliptic functions, 2009c. URL http://mathworld.wolfram.com/JacobiEllipticFunctions.html. The site was checked last on May 8th 2009.

mathworld. Complete elliptic integral of the first kind, 2009d. URL http://mathworld.wolfram.com/CompleteEllipticIntegraloftheFirstKind.html. The site was checked last on May 8th 2009.

Haruki Murakami. *A Wild Sheep Chase*. 1989. ISBN 4-7700-1706-5. Translated from the japanese 1982 original Hitsuji o meguru bōken by Alfred Birnbaum.

Mikio Nakahara. *Geometry, Topology and Physics*. Taylor and Francis Group, LLC, 2003. ISBN 978-0-7503-0606-5.

Charles Nash. *Differential Topology and Quantum Field Theory*. Academic Press, 1991, fourth edition 2003. ISBN 0-12-514076-2.

Charles Nash and Siddhartha Sen. *Topology and Geometry for Physicists*. Academic press inc. (London) ltd., 1983. ISBN 0-12-514080-0.

A. H. Castro Neto and Eduardo Fradkin. The thermodynamics of quantum systems and generalizations of zamolodchikov's c-theorem. *Nucl. Phys. B400*, pages 525–546, 1993.

Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 1995. ISBN 978-0-201-50397-5.

Finn Ravndal. *Lecture Notes on Statistical Physics Spring 1995*. Akademika AS UNIPUB, Oslo, Blindern, 1995.

Karl Rottmann. *Mathematische Formelsammlung, 4th edition,*. Bibliographisches Institut and F. A. Brockhaus, the Norwegian copy was published by Bracan Forlag, Nørhaven, Danmark, 1995. ISBN 82-7822-005-0. My copy is the 1995 Norwegian translation of this work, Matematisk Formelsamling.

Shinsei Ryu and Tadashi Takayanagi. Aspects of holographic entanglement entropy. *Journal of High Energy Physics 045*, 2006a.

Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from ads/cft. *arXiv:hep-th/0603001v2*, 2006b.

H. Saluer and M. Bauer. On some relations between local height probabilities and conformal invariance. *Nucl. Phys. B 320*, pages 591–624, 1989.

Hubert Saluer and Claude Itzykson. Two-dimensional field theories close to criticality. *Journal of Statistical Physics 48*, pages 449–475, 1987.

Bruno Schoeneberg. *Elliptic Modular Functions Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellung Band 203*. Springer-Verlag, Berlin Heidelberg New York, 1974. ISBN 0-387-06382-X. Translated from the German 1970 version by J. R. Smart and E. A. Schwandt.

Michael Spivak. *A Comprehensive Introduction to Differential Geometry Volume I*. Publish or Perish, INC., 1970, 3rd edition 2005. ISBN 0-914098-70-5.

Notto R. Thelle. *Zen - Lyden av én hånd*. Gyldendal Norsk Forlag AS, 2001. ISBN 82-05-27661-7.

wikipedia. Digamma function, 2009a. URL `http://en.wikipedia.org/wiki/Digamma_function`. The site was checked last on May 8th 2009.

wikipedia. Beta function, 2009b. URL `http://en.wikipedia.org/wiki/Beta_function`. The site was checked last on May 8th 2009.

wikipedia. Bernoulli polynomials, 2009c. URL `http://en.wikipedia.org/wiki/Bernoulli_polynomial`. The site was checked last on May 8th 2009.

wikipedia. Hurwitz zeta function, 2009d. URL `http://en.wikipedia.org/wiki/Hurwitz_zeta_function`. The site was checked last on May 8th 2009.

wikipedia. Bessel function, 2009e. URL `http://en.wikipedia.org/wiki/Bessel_function`. The site was checked last on May 8th 2009.

K. Wilson and J. Kogut. The renormalization group and the ϵ expansion. *Russ. transl., Mir, Moscow*, 1975.

Edward Witten. Anti de sitter space and hologrpaphy. *arXiv:hep-th/9802150v2*, 1998.

A. B. Zamolodchikov. "irreversibility" of the flux of the renormalization group in a 2d field theory. *JEPT Lett. 43 (1986)*, page 730, 1986.